Counting \((k,l)\)-sumsets in groups of prime order

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Abstract. A subset \(A\) of a group \(G\) is called \((k,l)\)-sumset, if \(A = kB - lB\) for some \(B \subseteq G\), where \(kB - lB = \{x_1 + \cdots + x_k - x_{k+1} - \cdots - x_{k+l} : x_1 , \ldots , x_{k+l} \in B\}\). Upper and lower bounds for the number \((k,l)\)-sumsets in groups of prime order are provided.

1 Introduction

Let \(p\) be a prime number and \(k, l\) be nonnegative integers with \(k + l \geq 2\). Write \(\mathbb{Z}_p\) for the group of residues modulo \(p\). A subset \(A \subseteq \mathbb{Z}_p\) is called \((k,l)\)-sumset, if \(A = kB - lB\) for some \(B \subseteq \mathbb{Z}_p\), where \(kB - lB = \{x_1 + \cdots + x_k - x_{k+1} - \cdots - x_{k+l} : x_1 , \ldots , x_{k+l} \in B\}\). Write \(SS_{k,l}(\mathbb{Z}_p)\) for the collection of \((k,l)\)-sumsets in \(\mathbb{Z}_p\).

B. Green and I. Ruzsa in [1] proved

\[
p^22^{p/3} \ll |SS_{2,0}(\mathbb{Z}_p)| \leq 2^{p/3+\Theta(p)}
\]

where \(\Theta(p)/p \to 0\) as \(p \to \infty\) and \(\Theta(p) \ll p(\log \log p)^2/(\log p)^{1/9}\) (hereafter logarithms are to base two).

The aim of this work is to obtain bounds for the number \(|SS_{k,l}(\mathbb{Z}_p)|\). We prove

**Theorem 1** Let \(p\) be a prime number and \(k, l\) be nonnegative integers with \(k + l \geq 2\). Then there exists a positive constant \(C_{k,l}\) such that

\[
C_{k,l}2^p/(2(k+l)-1) \leq |SS_{k,l}(\mathbb{Z}_p)| \leq 2^{p/(k+l+1)+(k+l-2)+o(p)}.
\]

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2 Definitions and auxiliary results

Let $\mathbb{R}$ be the set of real numbers, $f_i : \mathbb{Z}_p \to \mathbb{R}$, $i = 1, \ldots, m$, and $x \in \mathbb{Z}_p$. We set

$$ (f_1 * \cdots * f_m)(x) = \sum_{x_1 \in \mathbb{Z}_p} \cdots \sum_{x_m \in \mathbb{Z}_p} f_1(x_1) \cdots f_{m-1}(x_{m-1}) f_m(x - x_1 - \cdots - x_{m-1}) $$

(2)

and

$$ \hat{f}(x) = \sum_{y \in \mathbb{Z}_p} f(y) e^{2\pi i \frac{xy}{p}}. $$

The function $\hat{f}(x)$ is called Fourier transform of $f$.

**Lemma 2** We have

$$ (f_1 * \cdots * f_m)(x) = \hat{f}_1(x) \cdots \hat{f}_m(x). $$

(3)

**Proof.** By definition

$$ (f_1 * \cdots * f_m)(x) = \sum_{y \in \mathbb{Z}_p} (f_1 * \cdots * f_m)(y) e^{2\pi i \frac{xy}{p}} = $$

$$ = \sum_{y \in \mathbb{Z}_p} \sum_{y_1 \in \mathbb{Z}_p} \cdots \sum_{y_{m-1} \in \mathbb{Z}_p} f_1(y_1) \cdots f_{m-1}(y_{m-1}) \times $$

$$ \times f_m(y - y_1 - \cdots - y_{m-1}) e^{2\pi i \frac{y_1 x}{p}} \cdots e^{2\pi i \frac{y_{m-1} x}{p}} e^{2\pi i \frac{(y - y_1 - \cdots - y_{m-1}) x}{p}} = $$

$$ = \sum_{y_1 \in \mathbb{Z}_p} f_1(y_1) e^{2\pi i \frac{y_1 x}{p}} \cdots \sum_{y_{m-1} \in \mathbb{Z}_p} f_{m-1}(y_{m-1}) e^{2\pi i \frac{y_{m-1} x}{p}} \times $$

$$ \times \sum_{y \in \mathbb{Z}_p} f_m(y - y_1 - \cdots - y_{m-1}) e^{2\pi i \frac{(y - y_1 - \cdots - y_{m-1}) x}{p}} = \hat{f}_1(x) \cdots \hat{f}_m(x). $$

\[ \square \]

Denote the characteristic function of a set $A$ by $\chi_A(x)$. Let $A_1, \ldots, A_m$ be non-empty subsets of $\mathbb{Z}_p$. Then $(\chi_{A_1} * \cdots * \chi_{A_m})(x)$ will be the number of vectors $(x_1, \ldots, x_m) \in A_1 \times \cdots \times A_m$ such that $x \equiv x_1 + \cdots + x_m \pmod{p}$. Set $A_1 + \cdots + A_m = \{x_1 + \cdots + x_m \pmod{p} : x_1 \in A_1, \ldots, x_m \in A_m\}$. We define $S_{h,m}(A_1, \ldots, A_m) = \{x \in \mathbb{Z}_p : (\chi_{A_1} * \cdots * \chi_{A_m})(x) \geq h\}$, where $h > 0$. Further, for any integer $i$ and any $A \subseteq \mathbb{Z}_p$ denote the set $A + \cdots + A$ by $iA$, and the set $\{p - x : x \in A\}$ by $-A$. 
Theorem 3 (Cauchy-Davenport, [2]). Let $A_1, \ldots, A_m$ be non-empty subsets of $\mathbb{Z}_p$. Then $|A_1 + \cdots + A_m| \geq \min(p, |A_1| + \cdots + |A_m| - (m - 1))$.

Theorem 4 (Pollard, [3]). Let $A_1, A_2$ be non-empty subsets of $\mathbb{Z}_p$. Then

$$|S_{1,2}(A_1, A_2)| + \cdots + |S_{t,2}(A_1, A_2)| \geq t \min(p, |A_1| + |A_2| - t),$$

where $t \leq \min(|A_1|, |A_2|)$.

Theorems 3, 4 imply the following two statements.

Lemma 5 Let $A_1, \ldots, A_m$ non-empty subsets of $\mathbb{Z}_p$. Then

$$|S_{1,m}(A_1, \ldots, A_m)| + \cdots + |S_{t,m}(A_1, \ldots, A_m)| \geq t \min(p, |A_1| + \cdots + |A_m| - t - m + 2),$$

where $t \leq \min(|A_1|, \ldots, |A_m|)$.

Proof. Without loss of generality we assume $|A_1| = \min(|A_1|, \ldots, |A_m|)$. By Theorem 4 we have

$$|S_{1,2}(A_1, (A_2 + \cdots + A_m))| + \cdots + |S_{t,2}(A_1, (A_2 + \cdots + A_m))| \geq \min(p, |A_1| + |A_2 + \cdots + A_m| - t),$$

where $t \leq |A_1|$.

On the other hand by Theorem 3 we have

$$|A_2 + \cdots + A_m| \geq \min(p, |A_2| + \cdots + |A_m| - (m - 2)).$$

Substituting (5) in (4), we obtain

$$|S_{1,m}(A_1, \ldots, A_m)| + \cdots + |S_{t,m}(A_1, \ldots, A_m)| \geq |S_{1,2}(A_1, (A_2 + \cdots + A_m))| + \cdots + |S_{t,2}(A_1, (A_2 + \cdots + A_m))| \geq t \min(p, |A_1| + \cdots + |A_m| - t - m + 2).$$

□

Lemma 6 Let $A_1, \ldots, A_m$ be non-empty subsets of $\mathbb{Z}_p$ and $h \leq \min(|A_1|, \ldots, |A_m|)$. Then

$$|S_{h,m}(A_1, \ldots, A_m)| \geq \min(p, |A_1| + \cdots + |A_m| - m + 2) - 2(hp)^{1/2}.$$
Proof. Note that $|S_{i,m}(A_1, \ldots, A_m)| \geq |S_{i,m}(A_1, \ldots, A_m)|$ for $i \leq j$. Choose $h \leq t \leq \min (|A_1|, \ldots, |A_m|)$. By Lemma 5 we have
\[ t \min(p, |A_1| + \cdots + |A_m| - t - m + 2) \leq |S_{1,m}(A_1, \ldots, A_m)| + \cdots + |S_{i,m}(A_1, \ldots, A_m)| \leq hp + t|S_{h,m}(A_1, \ldots, A_m)|. \]
Putting $t = (hp)^{1/2}$, we get
\[ \min(p, |A_1| + \cdots + |A_m| - m + 2) - 2(hp)^{1/2} \leq \min(p, |A_1| + \cdots + |A_m| - m - (hp)^{1/2} + 2) - (hp)^{1/2} \leq |S_{h,m}(A_1, \ldots, A_m)|. \]

\[ \square \]

Lemma 7 Set $T_{r,s}(Z_p) = \{A \subset Z_p : |A| \leq p/(r+1)s\}$. Then there exists $s$ such that
\[ |T_{r,s}(Z_p)| \leq 2^{p/(r+1)}. \] (6)

Proof. Let $n, m$ be positive integers, $1 \leq m \leq n$. Then (see Lemma 6.8, [4])
\[ \sum_{0 \leq i \leq m} \binom{n}{i} \leq \left(\frac{en}{m}\right)^m. \] (7)

We choose $s$ such that
\[ es(r+1) \leq 2^s. \] (8)

Then by (7) we have (putting $n = p$ and $m = p/(r+1)s$)
\[ |T_{r,s}(Z_p)| = \sum_{0 \leq i \leq p/(r+1)s} \binom{p}{i} \leq (es(r+1))^{p/(r+1)s} \leq (2^s)^{p/(r+1)s} = 2^{p/(r+1)}. \]

\[ \square \]

Let $L$ be a positive integer. For each $y \in \{0, \ldots, p-1\}$ we define a partition $R_{y,L}$ of $Z_p$ on the intervals of the form $J_i^y = \{((iL + 1 + y) \mod p), \ldots, ((i+1)L + y) \mod p\}$, $0 \leq i \leq \lfloor p/L \rfloor - 1$. All intervals are $J_i^y$ of $R_{y,L}$ have length $L$, and the set $J_y = Z_p \setminus \bigcup_i J_i^y$ has cardinality $p - L \lfloor p/L \rfloor < L$. The set $J_y$ is called remainder partition $R_{y,L}$. In what follows we fix $y \in \{0, \ldots, p-1\}$ and consider the corresponding partition $R_{y,L}$. For every $A \subseteq Z_p$ and any integer $d$ define $d * A = \{da \mod p : a \in A\}$. The set $d * A$ is called dilation of $A$. The set $A \subseteq Z_p$ is called $L$-granular (see [1]), if some dilation of $A$ is a union of some of the intervals $J_i^y$ (other than remainder). We denote the family of $L$-granular subsets of $Z_p$ by $G_L(Z_p)$. 
Lemma 8 \ We have 
\[ |G_L(Z_p)| \leq p^{2p/L}. \]  
(9)

**Proof.** Denote the number of subsets of intervals (other than remainder) of the partition \( R_{y,L} \) of \( Z_p \) by \( g(R_{y,L}) \), and the number of different partitions \( R_{y,L} \) of \( Z_p \) by \( r(L) \). It is obvious that 
\[ |G_L(Z_p)| \leq g(R_{y,L}) r(L). \]  
(10)

Note that the number of intervals (other than remainder) of the partition \( R_{y,L} \) of \( Z_p \) is equal to \( \lfloor p/L \rfloor \), and the number of different partitions \( R_{y,L} \) of \( Z_p \) is at most \( p \). This and (10) imply the inequality (9).

Lemma 9 \ Let \( A \subseteq Z_p \) have size \( \alpha p \), and let \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) be positive real numbers and \( L > 0 \), \( k \), \( l \) be nonnegative integers satisfying \( k + l \geq 2 \). Suppose that 
\[ p > (\sqrt{8(k+1)L} \cdot 2^{2(k+l-1)} \cdot 2^{2(k+l-1)} \cdot 2^{2(k+l-1)} \cdot 2^{2(k+l-1)})^{1/2}. \]  
(11)

Then there exists a set \( A' \subseteq Z_p \) with the following properties:

(i) \( A' \) is \( L \)-granular;

(ii) \( |A \setminus A'| \leq \varepsilon_1 p \);

(iii) the set \( kA - lA \) contains all \( x \in Z_p \) for which 
\[ (\chi_{A'} \ast \cdots \ast \chi_{A'})^{k} \ast (-\chi_{A'})^{l}(x) \geq (\varepsilon_2 p)^{k+l-1}, \]  
with at most \( \varepsilon_3 p \) exceptions.

**Proof.** Let \( h \in \{0, \ldots, p-1\} \), and \( R_{h,L} \) be partition of \( Z_p \).

(i) For given set \( A \subset Z_p \) we define \( A' \subset Z_p \) as the union of intervals \( J_i^h \) of the partition \( R_{h,L} \), such that \( |A \cap J_i^h| \geq \varepsilon_1 L/2 \). From the definition it follows that \( A' \) is \( L \)-granular. It is easy to see that \( -(A) = -(A') \).

(ii) Let \( x \in A \setminus A' \). Then either \( x \in J_i \) or \( x \in A \cap J_i^h \) (i = 0, \ldots, \lfloor p/L \rfloor - 1), and \( |A \cap J_i^h| \leq \varepsilon_1 L/2 \). In the first case we have \( |J_i| < L \), and inequality (11) implies \( L \leq \varepsilon_1 p/2 \). Thus,
\[ |A \setminus A'| \leq \varepsilon_1 L \cdot 2^{p/L} \leq \varepsilon_1 p. \]

(iii) Let \( \hat{\chi}_A(x) \) be the Fourier transform of the characteristic function \( \chi_A \) of \( A \), so that 
\[ \hat{\chi}_A(x) = \sum_{y \in Z_p} \chi_A(y) e^{2\pi i yx/p} = \sum_{y \in A} e^{2\pi i yx/p}. \]
for all \( x \in \mathbb{Z}_p \). Take \( \delta = 4^{-k-1} \varepsilon_1^k \varepsilon_2^{k+1} \varepsilon_3^{1/2} \alpha^{-k-1} + 3/2 \), where \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) are from inequality (11). Set \( D = \{ x \neq 0 : |\hat{\chi}_A(x)| \geq \delta p \} \). We define the function \( f(x) \) as follows:

\[
f(x) = \frac{1}{2L-1} \sum_{j=-L+1}^{L-1} e^{2\pi i jx/p}.
\]

In the future we will show that there exists \( q \in \mathbb{Z}_p \setminus \{0\} \) such that for all \( x \in \mathbb{Z}_p \) it holds

\[
|\hat{\chi}_A(x)||1 - f^{k+1}(x)| \leq \delta p. \tag{12}
\]

The inequality (12) obviously holds for the case \( x = 0 \), since \( f(0) = 1 \), as well as for the case \( |\hat{\chi}_A(x)| \leq \delta p \), since \( f(x) \in [-1,1] \). Thus, it remains to show the existence of \( q \) such that the inequality (12) holds for all \( x \in D \). First we estimate the value of \( 1 - f(x) \). Denote by \( \langle x \rangle \) the distance from \( x \) to the nearest integer. We use the fact that \( 1 - \cos(2\pi x) \leq \frac{2\pi^2}{2L-1} \sum_{j=-L+1}^{L-1} \langle jx \rangle^2 \).

Recall that for \( |x| \leq 1 \)

\[
1 - x^m = (1 - x)(1 + x + x^2 + \cdots + x^{m-1}) \leq m(1 - x). \tag{14}
\]

From (13) and (14) it follows

\[
|\hat{\chi}_A(x)||1 - f^{k+1}(x)| \leq (k + 1)|\hat{\chi}_A(x)||1 - f(x)| \leq 8(k + 1)L^2 (\langle qx/p \rangle^2)^2 |\hat{\chi}_A(x)|.
\]

Note that if the inequality

\[
\langle qx \rangle^2 \leq \frac{1}{8(k + 1)L} \left( \frac{\delta p}{|\hat{\chi}_A(x)|} \right)^{1/2} \tag{15}
\]

holds for some \( q \in \mathbb{Z}_p \setminus \{0\} \) and for all \( x \in D \) then the inequality (12) also holds. Now we will prove that such \( q \) exists. By definition, we have

\[
\langle qx/p \rangle = \min((qx \pmod{p})/p, (p - qx \pmod{p})/p).
\]
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Set \(|D| = d\), \(D = \{r_1, \ldots, r_d\}\). We denote
\[ a_i = \left(1/\sqrt{8(k+1)L}(\delta p/|\hat{\chi}_A(r_i)|)^{1/2} \right). \]

Then the inequality (15) can be rewritten as
\[
\min\{q r_i \pmod{p}, p - q r_i \pmod{p}\} \leq p a_i, \quad \text{where} \quad i = 1, \ldots, d. \quad (16)
\]

Denote the set \(\{(x_1, \ldots, x_d) : x_1, \ldots, x_d \in Z_p\} \) by \(Z_p^d\). We split \(Z_p^d\) on disjoint subsets
\[
Z_p^d = \bigcup_{(i_1, \ldots, i_d)} Q_{i_1, \ldots, i_d},
\]
where
\[
Q_{i_1, \ldots, i_d} = \{(x_1, \ldots, x_d) : i_j p a_j < x_j \leq (i_j + 1)p a_j, j = 1, \ldots, d\}.
\]

Let \(\mu_d\) be number of different sets of \(Q_{i_1, \ldots, i_d}\). Using the fact that \(0 \leq i_j \leq 1/a_j - 1, j = 1, \ldots, d\), we have
\[
\mu_d \leq \prod_{i=1}^{d} \frac{1}{a_i}.
\]

Let us consider the following \(p - 1\) elements of \(Z_p^d\):
\[
(q r_1 \pmod{p}, \ldots, q r_d \pmod{p}), \quad \text{where} \quad r_1, \ldots, r_d \in D, \quad q = 1, \ldots, p - 1.
\]

We show that if
\[
p > \prod_{i=1}^{d} \frac{1}{a_i}, \quad (17)
\]
then there exists \(q\) such that for all \(r_i \in D, i = 1, \ldots, d\), the inequality (16) holds. We consider two cases:

(A) If \(\mu_d = p - 1\), then we take \(q = q_0\), where \(q_0 \in Z_p \setminus \{0\}\) such that \(q_0 r_1 \pmod{p}, \ldots, q_0 r_d \pmod{p}\) \(\in Q_{0, \ldots, 0}\).

(B) If \(\mu_d < p - 1\), then by pigeonhole principle, there are \(q_1, q_2 \in Z_p \setminus \{0\}\) such that the vectors \((q_1 r_1 \pmod{p}, \ldots, q_1 r_d \pmod{p})\) and \((q_2 r_1 \pmod{p}, \ldots, q_2 r_d \pmod{p})\) belong to the same set of \(Q_{i_1, \ldots, i_d}\). Obviously, when \(q = q_1 - q_2\) the inequality (16) holds.

We now show that inequality (17) is a consequence of (11). Indeed, by the Parseval’s identity, we have
\[
p^{-1} \left( \sum_{x \in D} |\hat{\chi}_A(x)|^2 + \sum_{x \in Z_p \setminus D} |\hat{\chi}_A(x)|^2 \right) = \sum_{x \in Z_p} |\chi_A(x)|^2 = \alpha p. \quad (18)
\]
From (18) it follows
\[ \sum_{x \in D} |\hat{\chi}_A(x)|^2 \leq \alpha p^2. \quad (19) \]

From (19) and the arithmetic and geometric mean inequality, we get
\[ \left( \prod_{x \in D} |\hat{\chi}_A(x)|^2 \right)^{1/d} \leq \frac{1}{d} \sum_{x \in D} |\hat{\chi}_A(x)|^2 \leq \frac{\alpha p^2}{d}. \]

i.e.
\[ \prod_{x \in D} |\hat{\chi}_A(x)| \leq \left( \frac{\alpha p^2}{d} \right)^{d/2}. \quad (20) \]

From (20) we get
\[ \left( \sqrt{8(k + l) L} \right)^d \left( \prod_{x \in D} \frac{|\hat{\chi}_A(x)|}{\delta p} \right)^{1/2} \leq \left( \sqrt{8(k + l) L} \alpha^{1/4} \delta^{-1/2} d^{-1/4} \right)^d. \quad (21) \]

It is easy to see that the right-hand side of (21) is an increasing function of \(d\) in the range \(d < 64(k + l)^2 L^4 \alpha / \delta^2 \).

On the other hand, from (19) we have \(d \delta^2 p^2 \leq \alpha p^2\). Hence, \(d \leq \alpha / \delta^2\). Consequently
\[ \left( \sqrt{8(k + l) L} \alpha^{1/4} \delta^{-1/2} d^{-1/4} \right)^d \leq \left( \sqrt{8(k + l) L} \right)^{\alpha / \delta^2}. \]

Recall that \(\delta = 4^{-(k+1)} \epsilon_1^{k+1} \epsilon_2^{k+1} \epsilon_3^{1/2} \alpha^{-(k+1)+3/2}\). From this it follows that there exists \(q\) such that the inequality (12) holds. Moreover, without loss of generality we can assume \(q = 1\) (this can be achieved by selecting an appropriate dilation of the set \(A\)).

Define two functions \(\chi_1(x)\) and \(\chi_2(x)\) as follows:
\[ \chi_1(x) = \frac{1}{|J|} (\chi_A * \chi_J)(x), \]
\[ \chi_2(x) = \frac{1}{|J|} (\chi_{-A} * \chi_J)(x), \]
where \(J = \{-(L-1), \ldots, L-1\}\). From (2) it follows that
\[ \chi_1(x) = \frac{1}{|J|} |A \cap (J + x)|, \quad (22) \]
\[ \chi_2(x) = \frac{1}{|J|} |(-A) \cap (J + x)|, \] (23)

and from (3) we have \( \widehat{\chi}_1(x) = \widehat{\chi}_{\alpha}(x)f(x) \) and \( \widehat{\chi}_2(x) = \widehat{\chi}_{-A}(x)f(x) \). Hence, by Parseval’s identity and from (3) we get

\[
\sum_{x \in \mathbb{Z}_p} \left| \left( \prod_{k=1}^{l} \chi_{-A} \right)(x) - \left( \prod_{k=1}^{l} \chi_{\alpha} \right)(x) \right|^2 = \]

\[
=p^{-1} \sum_{x \in \mathbb{Z}_p} \left| \left( \prod_{k=1}^{l} \chi_{-A} \right)(x) - \left( \prod_{k=1}^{l} \chi_{\alpha} \right)(x) \right|^2 = \]

\[
=p^{-1} \sum_{x \in \mathbb{Z}_p} \left| \widehat{\chi}_{-A}(x) \right|^2 \left| 1 - f^{k+1}(x) \right|^2 \leq \]

\[
\leq p^{-1} \left( \sup_{x \in \mathbb{Z}_p} \left| \widehat{\chi}_{\alpha}(x) \right|^2 \left| 1 - f^{k+1}(x) \right| \right)^2 \sum_{x \in \mathbb{Z}_p} \left| \widehat{\chi}_{\alpha}(x) \right|^2. \tag{24} \]

We have

\[
\left| \widehat{\chi}_{\alpha}(x) \right| = \left| \sum_{y \in \mathbb{Z}_p} \chi_{\alpha}(y) e^{2\pi i y x / p} \right| = \left| \sum_{y \in \mathbb{A}} e^{2\pi i y x / p} \right| \leq \sum_{y \in \mathbb{A}} \left| e^{2\pi i y x / p} \right| = \alpha p, \tag{25} \]

\[
\left| \widehat{\chi}_{-A}(x) \right| = \left| \sum_{y \in \mathbb{Z}_p} \chi_{-A}(y) e^{2\pi i y x / p} \right| = \left| \sum_{y \in -A} e^{2\pi i y x / p} \right| \leq \sum_{y \in -A} \left| e^{2\pi i y x / p} \right| = \alpha p. \tag{26} \]

From (12), (18), (24), (25) and (26) it follows

\[
\sum_{x \in \mathbb{Z}_p} \left| \left( \prod_{k=1}^{l} \chi_{-A} \right)(x) - \left( \prod_{k=1}^{l} \chi_{\alpha} \right)(x) \right|^2 \leq \]

\[
\leq \left( \sup_{x \in \mathbb{Z}_p} \left| \widehat{\chi}_{\alpha}(x) \right| \left| 1 - f^{k+1}(x) \right| \right)^2 \alpha^{2(k+1)-3} p^{2(k+1)-3} \leq \]
\[
\leq \alpha^{2(k+1) - 3} \delta^2 p^{2(k+1) - 1}.
\] (27)

Suppose that \(x \in A' (x \in -A')\). Then there exists an interval \(I\) of length \(L\) such that \(I \subseteq (x - (L - 1), \ldots, x + (L - 1))\) and \(x \in I\). From definition of \(A' (-A')\) it follows that \(|I \cap A| \geq \varepsilon_1 L/2 (|I \cap (-A)| \geq \varepsilon_1 L/2)\). From the definition of \(\chi_1(x) (\chi_2(x))\) it follows that \(\chi_1(x) \geq \varepsilon_1/4 (\chi_2(x) \geq \varepsilon_1/4)\). Observe, that \(\chi_1(x) \geq \varepsilon_1 \chi_{A'}(x)/4\) and \(\chi_2(x) \geq \varepsilon_1 \chi_{-A'}(x)/4\) hold for all \(x \in \mathbb{Z}_p\). From this and (2) it follows that

\[
(\chi_1 \ast \cdots \ast \chi_1 \ast \chi_2 \ast \cdots \ast \chi_2)(x) \geq \\
\geq \varepsilon_1^{k+1}(\chi_{A'} \ast \cdots \ast \chi_{A'} \ast \chi_{-A'} \ast \cdots \ast \chi_{-A'})(x)/4^{k+1}
\] (28)

for all \(x \in \mathbb{Z}_p\). In the case

\[
(\chi_{A'} \ast \cdots \ast \chi_{A'} \ast \chi_{-A'} \ast \cdots \ast \chi_{-A'})(x) \geq (\varepsilon_2 p)^{k+l-1},
\] (29)

by (28) we have

\[
(\chi_1 \ast \cdots \ast \chi_1 \ast \chi_2 \ast \cdots \ast \chi_2)(x) \geq \varepsilon_1^{k+1}(\varepsilon_2 p)^{k+l-1}/4^{k+1}.
\] (30)

Now we show that the number of elements \(x \in \mathbb{Z}_p\) such that satisfying (29)

and \((\chi_{A} \ast \cdots \ast \chi_{A} \ast \chi_{-A} \ast \cdots \ast \chi_{-A})(x) = 0\), does not exceed \(\varepsilon_3 p\). Denote the set of such elements by \(F\). Observe, that for every \(x \in F\)

\[
\|(\chi_{A} \ast \cdots \ast \chi_{A} \ast \chi_{-A} \ast \cdots \ast \chi_{-A})(x) - (\chi_1 \ast \cdots \ast \chi_1 \ast \chi_2 \ast \cdots \ast \chi_2)(x)\|^2 \geq \\
\geq \varepsilon_1^{2(k+1)} \varepsilon_2^{2(k+l-1)} p^{2(k+1-1)}
\] (31)

By (27) and (31)

\[
\alpha^{2(k+1) - 3} \delta^2 p^{2(k+1) - 1} \geq \\
\geq \sum_{x \in \mathbb{Z}_p} \|(\chi_{A} \ast \cdots \ast \chi_{A} \ast \chi_{-A} \ast \cdots \ast \chi_{-A})(x) - (\chi_1 \ast \cdots \ast \chi_1 \ast \chi_2 \ast \cdots \ast \chi_2)(x)\|^2 =
\]
Counting \((k, l)\)-sumsets in groups of prime order

\[\sum_{x \in F} \left| \chi_{A}^{*} \cdots \chi_{A}^{*} \chi_{-A}^{*} \cdots \chi_{-A}^{*}(x) - \chi_{1}^{*} \cdots \chi_{1}^{*} \chi_{2}^{*} \cdots \chi_{2}^{*}(x) \right|^2 + \]

\[\sum_{x \in (\mathbb{Z}/p\mathbb{Z})^\sim} \left| \chi_{A}^{*} \cdots \chi_{A}^{*} \chi_{-A}^{*} \cdots \chi_{-A}^{*}(x) - \chi_{1}^{*} \cdots \chi_{1}^{*} \chi_{2}^{*} \cdots \chi_{2}^{*}(x) \right|^2 \geq |F| \frac{\varepsilon_1^{2(k+1)} \varepsilon_2^{2(k+1-1)} p^{2(k+1-1)}}{4^{2(k+1)}} + \]

\[\sum_{x \in (\mathbb{Z}/p\mathbb{Z})^\sim} \left| \chi_{A}^{*} \cdots \chi_{A}^{*} \chi_{-A}^{*} \cdots \chi_{-A}^{*}(x) - \chi_{1}^{*} \cdots \chi_{1}^{*} \chi_{2}^{*} \cdots \chi_{2}^{*}(x) \right|^2 . \]

This implies

\[|F| \leq \frac{4^{2(k+1)} \alpha^{2(k+1)-3} \delta^2}{\varepsilon_1^{2(k+1)} \varepsilon_2^{2(k+1-1)} p^3 \varepsilon_3 p}. \]

\[\square\]

3 The proof of Theorem 1

3.1 The upper bound

Let \(k, l\) be nonnegative integers with \(k + l \geq 2\). Suppose that \(s\) satisfies \(e s(k + l + 1) \leq 2^s\). We divide a partition of \(SS_{k,l}(\mathbb{Z}_p)\) into two parts:

\[SS_{k,l}(\mathbb{Z}_p) = SS'_{k,l,s}(\mathbb{Z}_p) \cup SS''_{k,l,s}(\mathbb{Z}_p), \quad (32)\]

where

\[SS'_{k,l,s}(\mathbb{Z}_p) = \{B \in SS_{k,l}(\mathbb{Z}_p) : B = kA - lA \text{ and } |A| \leq p/(k + l + 1)s\}, \]

\[SS''_{k,l,s}(\mathbb{Z}_p) = \{B \in SS_{k,l}(\mathbb{Z}_p) : B = kA - lA \text{ and } |A| > p/(k + l + 1)s\}. \]

It is obvious that

\[|SS_{k,l}(\mathbb{Z}_p)| \leq |SS'_{k,l,s}(\mathbb{Z}_p)| + |SS''_{k,l,s}(\mathbb{Z}_p)|. \quad (33)\]

Since every set \(A \subseteq \mathbb{Z}_p\) generates one set of the form \(kA - lA\) we obtain

\[|SS'_{k,l,s}(\mathbb{Z}_p)| \leq |T_{k+1,s}(\mathbb{Z}_p)|. \quad (34)\]
By (7) and (34) we have
\[ |SS'_{\ell, s}(\mathbb{Z}_p)| \leq 2^{p/(k+l+1)}. \tag{35} \]

Now we prove an upper bound for $|SS'_{\ell, s}(\mathbb{Z}_p)|$. Suppose that the cardinality of $A \subseteq \mathbb{Z}_p$ is larger than $p/(k+l+1)s$. Let $p$ be a prime number such that for some nonnegative integers $k, l, L > 0$ and positive real numbers $\varepsilon_1, \varepsilon_2$ and $\varepsilon_3$ the condition (11) is fulfilled. By Lemma 9 there exists a subset $A'$ with properties (i) – (iii). We estimate the number of $(k, l)$-sumsets $kA - lA$ by counting pairs $(A', kA - lA)$.

Now let $A' \in G_{l}(\mathbb{Z}_p)$ be given. For any subset $A \subseteq \mathbb{Z}_p$ we denote by $\overline{A}$ the complement of the subset $A$ in $\mathbb{Z}_p$.

If $|A'| \geq p/(k + l + 1)$, then from (iii) of Lemma 9 we obtain that $kA - lA$ is a subset of the union of the set
\[ \overline{S}_{(\ell_{z} p)^{k+l-1}, k+l}^{l} \left( x_{A'}, x_{A'}', \ldots, x_{A'} \right) \]
and a set of cardinality not exceeding $\varepsilon_3 p$. By Lemma 6 we have
\[ |S_{(\ell_{z} p)^{k+l-1}, k+l}^{l} \left( x_{A'}, x_{A'}', \ldots, x_{A'} \right)| \geq \min(p, (k + l)|A'| - (k + l) + 2) - 2((\ell_{z} p)^{k+l-1} - 1)^{1/2}. \]
If $|A'| \geq p/(k + l + 1)$, we obtain
\[ |S_{(\ell_{z} p)^{k+l-1}, k+l}^{l} \left( x_{A'}, x_{A'}', \ldots, x_{A'} \right)| = \]
\[ = p - |S_{(\ell_{z} p)^{k+l-1}, k+l}^{l} \left( x_{A'}, x_{A'}', \ldots, x_{A'} \right)| \leq \]
\[ \leq p/(k + l + 1) + 2\ell_{z}^{(k+l-1)/2}p^{(k+l-1)/2} + (k + l - 2). \]

It is obvious that for any subset $B \subseteq \mathbb{Z}_p$ the set $kB - lB$ uniquely determines the set $kB - lB$. From above it follows that the number of choices $kA - lA$ for given $A'$ of cardinality exceeding $p/(k + l + 1)$, is at most
\[ 2^{p/(k+l+1)+(k+l+2) - (2\ell_{z}^{(k+l-1)/2}p^{(k+l-1)/2} + \varepsilon_3)p}. \tag{36} \]

If $|A'| < p/(k + l + 1)$, then by (i) of Lemma 9 we have $|A \setminus A'| \leq \varepsilon_1 p$. This implies that $|A| \leq |A'| + \varepsilon_1 p$. Since every set $A \subseteq \mathbb{Z}_p$ generates exactly one set of form $kA - lA$, we obtain that the number of choices $kA - lA$ for given $A'$ of cardinality not exceeding $p/(k + l + 1)$, is at most
\[ 2^{p/(k+l+1)+\varepsilon_1 p}. \tag{37} \]
From (36), (37), Lemma 8 by applying Lemma 9 with parameters \( \varepsilon_1 = \varepsilon_3 = \varepsilon \), \( L = 1 + \lfloor 1/\varepsilon \rfloor \) and \( \varepsilon_2 = \varepsilon^{2/(k+1-1)p(2-k-1)/(k+1-1)} \), we obtain

\[
|SS'_{k,1,s}(Z_p)| \leq 2^{p/(k+1+1) + (k+1-2) + o(p)}.
\]

(38)

From (33), (35) and (38) it follows that

\[
|SS_{k,1}(Z_p)| \leq 2^{p/(k+1+1)} + 2^{p/(k+1+1) + (k+1-2) + o(p)} = 2^{p/(k+1+1) + (k+1-2) + o(p)}.
\]

3.2 The lower bound

Set \( SS_{k,1}(Z_p, \mathbb{P}) = \{ A : \mathbb{P} \subseteq A, A \in SS_{k,1}(Z_p) \} \) and \( L = \lfloor p/(2(k+1) - 1) \rfloor - 1 \).

Lemma 10 Let \( k, l \) be nonnegative integers with \( k + l \geq 2 \), and let \( \mathbb{P} \subseteq Z_p \) be arbitrary arithmetic progression of length \( (k+1)(L-1) + 1 \). Then there exists a positive constant \( C_{k,1} \) such that

\[
|SS_{k,1}(Z_p, \mathbb{P})| \geq C_{k,1} 2^{p/(2(k+1)-1)}.
\]

Proof. Without loss of generality we assume \( \mathbb{P} = \{ k - 1k, \ldots, kL - l \} \). All of our sets will be of the form

\[
A = A(B) = k(B \cup \{ -(2L + 1), 2L + 1 \}) - l(B \cup \{ -(2L + 1), 2L + 1 \}),
\]

where \( B \subseteq \{ -L, -L + 1, \ldots, L \} \) and \( -B = B \). It is easy to see that different sets \( B \subseteq \{ -L, -L + 1, \ldots, L \} \) generate different sets \( A(B) \).

Set \( N_{k,1} = \lceil \log (8(k + 1)^2) / \log (4/3) \rceil \) and

\[
X = \{ 0, 1, \ldots, N_{k,1} \} \cup \bigcup_{i=1}^{k+1-1} \{ \lfloor (i + 1)L/(k + 1) \rfloor - N_{k,1}, \ldots, \lfloor (i + 1)L/(k + 1) \rfloor \}.
\]

We define the set \( B \subseteq \{ -L, -L + 1, \ldots, L \} \) as follows:

\[
B = B(C) = -C \cup C \cup X \cup -X,
\]

where elements of the set \( C \) are picked from the set \( \{ 1, \ldots, L \} \setminus X \) randomly, independently, with probability \( 1/2 \). Set

\[
Y = \{ 0 \} \cup \{ k + 1, \ldots, (k + 1)N_{k,1} \} \cup \bigcup_{i=1}^{k+1-1} \{ (i + 1)L - (k + 1)N_{k,1}, \ldots, (i + 1)L \}.
\]
It is obvious that \(-Y \cup Y \subseteq kB - lB\). If \(x \not\in kB - lB\), then in the representation \(x\) in the form \(x = x_1 + \cdots + x_k - x_{k+1} - \cdots - x_{k+l}\), there exists at least one \(x_i\) \((i \in \{1, \ldots, k + l\})\) such that \(x_i \not\in B\). Set

\[
Q(x) = \{ (x_1, \ldots, x_{k+l}) : x = \sum_{i=1}^{k} x_i - \sum_{j=k+1}^{k+l} x_j, x_1, \ldots, x_{k+l} \in \{-L, \ldots, L\}\}
\]

and suppose that \(|Q(x)| = q\).

We say that the vectors \((x_1, \ldots, x_{k+l})\) and \((y_1, \ldots, y_{k+l})\) do not intersect, if \([x_1, \ldots, x_{k+l}] \cap [y_1, \ldots, y_{k+l}] = \emptyset\).

Set \(R_0 = \{(k + l)N_{k+1} + 1, \ldots, L\}\). We show that for every \(x \in -R_0 \cup R_0\) the following inequality

\[
\Pr(x \not\in kB - lB) \leq \left(\frac{3}{4}\right)^{\frac{|x|}{k+l}}
\]

(39)

holds. We have

\[
\Pr(x \not\in kB - lB) = \\
= \Pr([x_1^1 + \cdots + x_{k+1}^1 - \cdots - x_{k+l}^1 \not\in kB - lB] \& \ldots \\
\ldots \& (x_1^q + \cdots + x_{k+1}^q - \cdots - x_{k+l}^q \not\in kB - lB)) \leq \\
\leq \Pr(([x_1^{11} + \cdots + x_{k+1}^{11} - \cdots - x_{k+l}^{11} \not\in kB - lB] \& \ldots \\
\ldots \& (x_1^{1n} + \cdots + x_{k+1}^{1n} - \cdots - x_{k+l}^{1n} \not\in kB - lB)) = \\
= \Pr \left((x_1^{11} \not\in B \lor \cdots \lor x_{k+1}^{11} \not\in B) \& \ldots \& (x_1^{1n} \not\in B \lor \cdots \lor x_{k+1}^{1n} \not\in B) \right) = \\
= \Pr \left((x_1^{11} \not\in B) \lor \cdots \lor (x_{k+1}^{11} \not\in B) \right) \cdots \Pr \left((x_1^{1n} \not\in B) \lor \cdots \lor (x_{k+1}^{1n} \not\in B) \right) = \\
= \Pr \left((x_1^{11} \in B) \& \ldots \& (x_{k+1}^{11} \in B) \right) \times \ldots \\
\ldots \times \Pr \left((x_1^{1n} \in B) \& \ldots \& (x_{k+1}^{1n} \in B) \right) = \\
= \left(1 - \Pr \left((x_1^{11} \in B) \& \ldots \& (x_{k+1}^{11} \in B) \right) \right) \times \ldots \\
\ldots \times \left(1 - \Pr \left((x_1^{1n} \in B) \& \ldots \& (x_{k+1}^{1n} \in B) \right) \right),
\]

(40)

where the vectors \((x_i^1, \ldots, x_{k+1}^i) \in Q(x), i = 1, \ldots, q\), and the vectors \((x_1^{j1}, \ldots, x_{k+1}^{jn})\), \(j = 1, \ldots, n \leq q\), are pairwise disjoint.
Note that the vectors \( (x-i(k+l-1), i, \ldots, i, -i, \ldots, -i) \) are pairwise disjoint for every \( x \in \mathbb{R} \), where \(-\lfloor |x|/(k+l) \rfloor \leq i \leq -1\), and \( x \in \mathbb{R} \), where \( 1 \leq i \leq \lfloor x/(k+l) \rfloor \). From this and (40) we obtain the inequality (39).

Set \( \mathcal{L}_j = \{ jL + 1, \ldots, (j+1)L - (k+l)N_{k,l} - 1 \}, j = 1, \ldots, k+l - 1 \). Similarly to the inequality (39) we have

\[
\Pr(x \notin kB - lB) \leq \left( \frac{3}{4} \right)^{\left\lfloor \frac{j+1-k-L}{k+l} \right\rfloor},
\]

(41)

where \( x \in -\mathcal{L}_j \cup \mathcal{L}_j, j = 1, \ldots, k+l - 1 \).

From (39) and (41) it is easy to see that

\[
\Pr(P \notin kB - lB) \leq (k+l)^{\sum_{x \geq (k+l)N_{k,l} - 1} \left( \frac{3}{4} \right)^{\left\lfloor \frac{x+1}{k+l} \right\rfloor}}.
\]

(42)

Note that if \( N_{k,l} \geq \log \left( 8(k+l)^2 \right)/\log \left( 4/3 \right) \), the right-hand side of (42) does not exceed \( 1/2 \). This leads that there exists at least \( 2L^{(k+l)N_{k,l} - 1} \) subsets \( B \subseteq \{ -L, -L + 1, \ldots, L \} \) such that \( P \subseteq kB - lB \). \( \square \)

Let \( k,l \) be nonnegative integers with \( k+l \geq 2 \), and let \( P \subseteq \mathbb{Z}_p \) be arbitrary arithmetic progression of length \( (k+l)(L-1) + 1 \). By Lemma 10 we have

\[
|SS_{k,l}(\mathbb{Z}_p)| \geq |SS_{k,l}(\mathbb{Z}_p, P)| \geq C_{k,l}2^{p/(2(k+l)-1)}.
\]

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References


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