Parallel enumeration of degree sequences of simple graphs II

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Abstract. In the paper we report on the parallel enumeration of the degree sequences (their number is denoted by $G(n)$) and zerofree degree sequences (their number is denoted by $(G_z(n))$) of simple graphs on $n = 30$ and $n = 31$ vertices. Among others we obtained that the number of zerofree degree sequences of graphs on $n = 30$ vertices is $G_z(30) = 5876236938019300$ and on $n = 31$ vertices is $G_z(31) = 22974847474172374$. Due to Corollary 21 in [52] these results give the number of degree sequences of simple graphs on 30 and 31 vertices.

1 Introduction

In the practice an often occurring problem is the ranking of different objects (examples can be found e.g. in [52]), assigning points to the objects and then ranking of the objects on the base of the sum of the assigned to them points.

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Especially extensive bibliography has the case when the results are represented by a simple graph and the problem is the test, reconstruction and enumeration of the degree sequences. Havel in 1955 [42], Erdős and Gallai in 1960 [16, 32, 77], Hakimi in 1962 [39], Knuth in 2008 [61], Tripathi et al. in 2010 [89] proposed a method to decide, whether a sequence of nonnegative integers can be the degree sequence of a simple graph. Sierksma and Hoogeveen in 1991 [83] compared seven known methods. The running time of their algorithms in worst case is $\Omega(n^2)$. In 2007 Takahashi [86], in 2009 Hell and Kirkpatrick [43], in 2011 Iványi et al. [52] and in April of 2012 Király [58] proposed an algorithm, whose worst running time is $\Theta(n)$.

There are several new proofs for the classical Havel-Hakimi and Erdős-Gallai theorems [26, 32, 63, 70, 75, 87, 88, 89].

Extensions of the algorithms for $(0,b)$-graphs [8, 9, 24, 23, 25, 27, 69, 75, 90, 92] and $(a,b)$-graphs [44, 45, 46, 53] are also known.

As an application of our linear time algorithm we describe Havel-Hakimi-Enumerative algorithm (EGE) and its parallel version used to enumerate the different degree sequences of simple graphs for 30 vertices. We also present the linear test version of Havel-Hakimi algorithm (HHL).

Let $n \geq 1$. We call a sequence $s = (s_1, \ldots, s_n)$ $(l,u,n)$-bounded, if $0 \leq s_i \leq n$ for $i = 1, \ldots, n$, $n$-bounded, if it is $(0,n-1,n)$-bounded, $n$-regular, if the conditions $n-1 \geq s_1 \geq \cdots \geq s_n \geq 0$ hold, and $n$-even, if the sum of the elements of $s$ is even. If there exists a graph with $n$ vertices which has the degree sequence $s$, then we say that $s$ is $n$-graphical. If such graph does not exist, then we say that $s$ is nongraphical. A sequence is zerofree, if it does not contain zero. If $n$ is not necessary, then we omit it in the terms $n$-bounded, $n$-regular, $n$-even and $n$-graphical. The first $i$ elements of an $n$-regular $s$ are called the head, and the last $n-i$ elements are called the tail, belonging to the element $i$ of $s$.

2 Earlier results

A classical problem of the graph theory is the enumeration of the sorted degree sequences of different graphs—among others simple graphs. For example The On-Line Encyclopedia of Integer Sequences contains for $n = 1, \ldots, 29$ vertices the number of degree sequences of simple graphs (the values for $n = 20, \ldots, 23$ were set in July of 2011 by Nathann Cohen [28], and for $24, \ldots, 29$ in 15 November, 2011 by us [48, 52]) and the number of zerofree degree sequences of simple graphs (the values for $n = 1, \ldots, 9$ were set in 12 June, 2004 by
N. J. Sloane, for $n = 10, \ldots, 20$ in 12 August, 2006 by Gordon Royle, for $n = 21, 22, \text{ and } 23$ in August 31, 2011, and in December 10, 2012 by Frank Ruskey [80], and the values for $n = 24, \ldots, 29$ by us [50, 51].

In this section we review the theoretical and practical results connected with the enumeration of simple graphs.

2.1 Exact enumeration results

It is known [52, equation (23)] that if $n \geq 1$, then the number $R(n)$ of the regular sequences is

$$ R(n) = \binom{2n-1}{n} $$

(1)

and the number $R_z(n)$ of the zerofree regular sequences is [52, equation (24)]

$$ R_z(n) = \binom{2n-2}{n} $$

(2)

implying [52]

$$ \lim_{n \to \infty} \frac{R(n+1)}{R(n)} = \lim_{n \to \infty} \frac{R_z(n+1)}{R_z(n)} = 4 $$

(3)

and

$$ \lim_{n \to \infty} \frac{R_z(n)}{R(n)} = \frac{1}{2}, $$

(4)

and

$$ R(n) = \frac{4^n}{2 \sqrt{\pi n}} + O \left( \frac{4^n}{n^{3/2}} \right) $$

(5)

and

$$ R_z(n) = \frac{4^n}{4 \sqrt{\pi n}} + O \left( \frac{4^n}{n^{3/2}} \right). $$

(6)

Table 1 in [52] shows the values values of $R(n)$ for $n = 1, \ldots, 38$, Table 4 in [51] for $n = 39, \ldots, 60$, and in [47, 51, 68] the values are presented for $n = 1, \ldots, 1200$. Table 1 in Subsection 3.3 presents the values $R(n)/R(n+1)$ for $n = 1, \ldots, 32$ and [68] for $n = 1, \ldots, 1200$.

Figure 1 in Subsection 3.3 shows the values of $R_z(n)/R_z(n+1)$ for $n = 1, \ldots, 32$.

In 1987 Ascher derived the following explicit formula for the number $E(n)$ of even sequences.
Lemma 1 (Ascher [2], Sloane and Pfoffe [85]) If $n \geq 1$, then the number of even sequences $E(n)$ is

$$E(n) = \frac{1}{2} \left( \binom{2n-1}{n} + \binom{n-1}{\lceil n/2 \rceil} \right).$$

(7)

Proof. See [2]. □

Table 1 in [52] contains the values of $E(n)$ and $E(n+1)/E(n)$ for $n = 1, \ldots, 31$.

(7) implies (see [52])

$$\lim_{n \to \infty} \frac{E(n+1)}{E(n)} = 4$$

(8)

and

$$E(n) = \frac{4^n}{8\sqrt{\pi n}} + O\left( \frac{4^n}{n^{3/2}} \right).$$

(9)

further (1) and (7) imply

$$\lim_{n \to \infty} \frac{E(n)}{R(n)} = \frac{1}{2},$$

(10)

(2) and (7) imply

$$\frac{R_z(n)}{E(n)} = \frac{2n - 2}{2n - 1} = 1 - \frac{1}{2n - 1} \quad \text{and} \quad \lim_{n \to \infty} \frac{R_z(n)}{E(n)} = 1.$$ (11)

Table 1 in [52] shows the values of $E(n)$ for $n = 1, \ldots, 38$, Table 4 in [51] for $n = 39, \ldots, 60$, the list of [64] for $n = 1, \ldots, 1000$, and [68] for $n = 31, \ldots, 1200$.

Figure 3 in [52] shows the values of $E_z(n)$ for $n = 1, \ldots, 20$, and [68] $n = 1, \ldots, 1200$. Table 5 in [51] shows the values of $E_z(n/R(n))$ for $n = 1, \ldots, 750$ (see [52, 68]). Recently Librandi [64] published the values of $E(n)$ up to $n = 1000$ and we continued the computations up to 1200 [51, 68].

The following theorem gives a very useful connection between the values of $G(n)$ and $G_z(n)$: it helped to decrease the computing time of $G(29)$ with about 50%.

Lemma 2 (Íványi, Lucz, Móri, Sótér [52]) If $n \geq 2$, then the number of $n$-graphical sequences $G(n)$ can be computed from the number of $(n-1)$-graphical sequences $G(n-1)$ and the number of $n$-graphical zero-free sequences $G_z$:

$$G(n) = G(n-1) + G_z(n),$$
and if \( n \geq 1 \) then
\[
G(n) = 1 + \sum_{i=2}^{n} G_{\mathcal{Z}}(i).
\]

**Proof.** If an even sequence \( s = (s_1, \ldots, s_n) \) contains at least one zero, then \( s_n = 0 \) and \( s' = (s_1, \ldots, s_{n-1}) \) is graphical or not. If \( a = (a_1, \ldots, a_{n-1}) \) is \((n-1)\)-graphical, then \( a' = (a_1, \ldots, a_{n-1}, 0) \) is \( n \)-graphical.

The set of the \( n \)-graphical sequences \( \mathcal{S}(n) \) consists of two subsets: set of zerofree sequences \( \mathcal{S}_{\mathcal{Z}}(n) \) and the set of the remaining sequences \( \mathcal{S}_0(n) \). There is a bijection between the set of the \((n-1)\)-graphical sequences and such \( n \)-graphical sequences, which contain at least one zero. Therefore
\[
|\mathcal{S}| = |\mathcal{S}_{\mathcal{Z}}| + |\mathcal{S}_0| = G_{\mathcal{Z}}(n) + G(n-1).
\]

Using the parallel version EGP (see the next section) of EGE we computed \( G(n) \) up to \( n = 29 \). These numbers can be found in Table 2 of [52].

**Theorem 3** (Burns [22]) There exist positive constants \( c \) and \( C \) such that the following bounds of the function \( G(n) \) are true for \( n \geq 1 \):
\[
\frac{4n}{cn} < G(n) < \frac{4n}{(\log n)^c \sqrt{n}}.
\]

**Proof.** See [22].

This result implies that the asymptotic density of the graphical sequences is zero among the even sequences.

**Corollary 4** If \( n \geq 1 \), then there exists a positive constant \( C \) such that
\[
\frac{G(n)}{E(n)} < \frac{1}{(\log_2 n)^C}
\]
and
\[
\lim_{n \to \infty} \frac{G(n)}{E(n)} = 0.
\]

**Proof.** (13) is a direct consequence of (7) and (12).

Table 1 in [52] contains the values of \( G(n) \) and \( G(n+1)/G(n) \) for \( n = 1, \ldots, 29 \). Table 5 in [51] contains values of \( G_{\mathcal{Z}}(n) \), \( G_{\mathcal{Z}}(n)/R(n) \), and \( G(n)/R(n) \) for \( n = 1, \ldots, 29 \).

We remark that a zerofree degree sequence belongs to a graph not containing isolated vertex, therefore the number of zerofree graphical degree sequences
$G_z(n)$ is at the same time also the number of degree sequences of simple graphs, not containing isolated vertex.

There are several classic asymptotic results, e.g. due to Bender and Canfield [7], Bollobás [17, 18, 19], Harary and Palmer [41], Kleitman and Winston [56, 60], Reid [78], Winston and Kleitman [91]. A modern direction is to get approximate results by sampling of random graphs (see e.g. the papers of Erdős, Király and Miklós [34], further of Miklós, Erdős and Soukop [?].

An interesting connected problem is the characterization of pairs of different directed graphs having a pair of prescribed indegree and outdegree sequences [8, 9, 10, 11, 12, 14, 15, 20, 40, 72, 76, 81].

Another interesting related questions are the unicity of the realizations of the degree sequences [29, 55, 62, 82] and the parallel realization of degree sequences [1].

Several recent papers consider the problem of approximate enumeration of the number of all realizations of simple graphs (see e.g. [13, 34, 35, 36, 37, 38, 59, 71]). In 1978 Bender and Canfield [7] characterized the asymptotic number of realizations of given graphical degree sequences, while in 2012 Zoltán Király [58] proposed an algorithm which with polynomial delay lists all realizations of a given graphical sequence.

2.2 Earlier algorithmic results

In this subsection the linear Havel-Hakimi algorithm (HHL) based on Havel-Hakimi theorem [39, 42] and the enumerating Erdős-Gallai algorithm (EGE) based on Erdős-Gallai theorem [32] are shortly described.

2.2.1 Linear Havel-Hakimi algorithm (HHL)

In a previous paper [52] we described the classical Havel-Hakimi [39, 42] and Erdős-Gallai [32] algorithms and their some improvements as linear Erdős-Gallai (EGL) and jumping Erdős-Gallai (EGLJ) algorithms.

It is worth to remark that our linear Erdős-Gallai algorithm is applied in the solution of different problems connected with degree sequences [5, 6, 21, 31].

Here we present the linear version of Havel-Hakimi algorithm (HHL) [46] and compare it with the previous linear algorithms EGL and EGLJ [52]. It is important to remark that this linear version of HH only tests the investigated sequences without their reconstruction.

In the worst case the original Havel-Hakimi algorithm requires quadratic time to test the $(0,1,n)$-regular sequences. Using the new concepts weight
point and reserve we reduced the worst running time to \( O(n) \).

Let \( s = (s_1, \ldots, s_n) \) be a potential graphical sequence. The definition of the \emph{weight point} \( w_i \) belonging to \( s_i \) was introduced in \cite{52} in connection with Erdős-Gallai-Linear: if \( s_1 \geq i \), then \( w_i \) is the largest \( k \) (\( 1 \leq k \leq n \)) having the property \( s_k \geq i \). But if \( s_1 < i \), then \( w_i = 0 \). EGL exploits the property \( w_i \) ensuring that if \( i \leq w_i \), then the key expression \( \min_j s_k \) in the Erdős-Gallai theorem equals \( i \), otherwise equals \( s_k \).

In HHL the weight point \( w_i \) determines the increment of the tail capacity when we switch to the investigation of the next element of \( s \).

The reserve \( r_i \) belonging to \( s_i \) is defined as the unused part of the actual tail capacity and can be computed by the formulas

\[
r_1 = w_1 - 1 - s_1 \quad (15)
\]

and

\[
r_i = w_i + r_{i-1} - s_i \quad \text{for} \quad 2 \leq i \leq n - 1. \quad (16)
\]

**Theorem 5** The running time of Havel-Hakimi-Linear is in best case \( \Theta(1) \), and in worst case it is \( \Theta(n) \).

**Proof.** If the condition in line 1 or 3 holds, then the running time is \( \Theta(1) \). If not, then we decrease the actual \( w \) at most \( n \) times and the remaining operations require \( O(1) \) operations for all reductions. \( \square \)

### 2.2.2 Enumerating Erdős-Gallai algorithm (EGE)

A classical problem of the graph theory is the enumeration of the degree sequences of different graphs—among others simple graphs. For example The On-Line Encyclopedia of Integer Sequences \cite{84} contains for \( n = 1, \ldots, 30 \) vertices the number of degree sequences of simple graphs (the values for \( n = 20, \ldots, 23 \) were set in July of 2011 by Nathann Cohen, in November 15, 2011 for \( 24, \ldots, 29 \) and in 29 July of 2013 for \( n = 30 \) by us \cite{48}).

We applied the new quick EGL to get these numbers for larger values of \( n \).

Our starting point was to test all regular sequences and so to enumerate the graphical ones. Equation 1 gives the number of regular sequences.

According to Erdős-Gallai theorem \cite{32} the sum of the elements of a graphical sequence is always even. Therefore it is sufficient to test only the even sequences. In 1987 Ascher \cite{2} derived Lemma 1, containing an explicit formula for the number of even sequences \( E(n) \).

According to Lemma 2 it is enough to test only the zero-free even sequences.
This lemma was the base of Erdős-Gallai Enumerative algorithm (EGE) used to enumerate the graphical sequences for \( n = 23, \ldots, 29 \) [51].

We enumerated the graphical sequences of simple graphs on \( n = 30 \) and 31 vertices using algorithm EGE2. The running time of EGE was substantially (with about 30 \%) decreased due to Lemma 9.

We prepare the enumeration of degree sequences of simple graphs on 32 vertices. The running time of EGE2 would be about 320 years for a computer with one processor having 2.2 GHz speed. We wish to decrease the running time of EGE2 using Lemmas 10 and 11.

### 2.3 Earlier simulation results

The papers [44, 45, 46, 51, 52, 66] and OEIS [64, 73, 74] contain many simulation results. We describe them together with the new results in Subsection 3.3.

It is worth to mention other methods of enumeration of graph sequences as generation of random graphs (e.g. [65] and generation of graphical partitions (see e.g. [3, 4, 30, 33]).

### 3 New results

In this section we describe the new mathematical and simulation results.

#### 3.1 New enumerative results

At first we give a new formula for the number of zerofree even sequences. This formula is more sophisticated than Ascher’s formula, and its application requires more time, but it has the advantage that we can extend it to a formula for \( E_z(n) \). Let \( s \) be an \( n \)-even sequence and let \( s' = (s'_1, \ldots, s'_n) \) be defined by \( s'_i = s_i + n - i \) for \( i = 1, \ldots, n \). Then the number of different possible sequences \( s \) is \( E(n) \) and the number of different sequences \( s' \) is \( R(n) \).

If \( j = 0, 1, 2, \) or 3 and \( n = 4k + j \), then let \( E(n) \) be denoted by \( E(k, j) \).

**Lemma 6** If \( n \geq 1 \) and \( n = 4k + j \), then

\[
E(k, 0) = \sum_{i=0}^{2k-1} \binom{4k-1}{2i} \binom{4k}{4k-2i},
\]

\[
E(k, 1) = \sum_{i=0}^{2k} \binom{4k}{2i} \binom{4k+1}{4k-2i+1},
\]  

(17)  

(18)


\[ E(k, 2) = \sum_{i=0}^{2k} \binom{4k + 1}{2i + 1} \binom{4k + 2}{4k - 2i + 1}, \quad (19) \]

\[ E(k, 3) = \sum_{i=0}^{k} \binom{4k + 2}{2i + 1} \binom{4k + 3}{4k - 2i + 2}. \quad (20) \]

**Proof.** Let

\[ \sum_{i=1}^{n} s_i = S(s) \quad \text{and} \quad \sum_{i=1}^{n} s'_i = S'(s). \quad (21) \]

According to the value of \( j \) we consider four cases. Since \( s \) is an even sequence, therefore \( S(s) \) is even in all cases.

1. If \( j = 0 \), then

\[ S'(s) = S(s) + \sum_{i=0}^{4k-1} i = S(s) + 2k(4k - 1), \quad (22) \]

and so \( S(s') \) is also even, therefore it contains an even number of odd elements. The interval \([0, 8k - 2]\) contains \( 8k - 1 \) elements and among them \( 4k \) even and \( 4k - 1 \) odd elements, so for \( s' \) we can choose \( 2i \) odd elements from \( 4k - 1 \) candidates and \( 4k - 2i \) even elements from \( 4k + 1 \) candidates, so

\[ E(k, 0) = \sum_{i=0}^{2k-1} \binom{4k - 1}{2i} \binom{4k}{4k - 2i}. \quad (23) \]

2. If \( j = 1, 2 \) or \( j = 3 \), then the proof is similar to the proof in the first case.

\[ \square \]

For example let \( n = 4 \), then \( k = 1, j = 0 \) and

\[ E(4) = E(1, 0) = \sum_{i=0}^{1} \binom{3}{2i} \binom{4}{4 - 2i} = 1 \cdot 1 + 3 \cdot 6 = 19. \quad (24) \]

As another example let \( n = 6 \), then \( k = 1, j = 2 \) and

\[ E(6) = \sum_{i=0}^{2} \binom{5}{1} \binom{6}{3} + \binom{5}{5} \binom{6}{1} = 530 + 200 + 6 = 236. \quad (25) \]
Let the number of zerofree even sequences denoted by \( E(z)(n) \). Let \( q = (q_1, \ldots, q_n) \) be a zerofree \( n \)-even sequence and let \( q' = (q'_1, \ldots, q'_n) \) be defined by \( q'_i = q_i + n - i \) for \( i = 1, \ldots, n \). Then the number of different possible sequences \( q \) is \( E(z)(n) \) and the number of different sequences \( q' \) is \( R(z)(n) \).

**Theorem 7** Let \( n = 4k + j \) for \( k = 0, 1, \ldots \) and \( j = 0, 1, 2, 3 \), further let \( E(z)(n) \) be denoted by \( E(z)(k,j) \). Then

\[
E_z(k,0) = \sum_{i=0}^{2k-1} \left(\begin{array}{c} 4k-1 \\ 2i \end{array}\right) \left(\begin{array}{c} 4k-1 \\ 4k-2i \end{array}\right),
\]

\[
E_z(k,1) = \sum_{i=0}^{2k} \left(\begin{array}{c} 4k \\ 2i \end{array}\right) \left(\begin{array}{c} 4k \\ 4k-2i+1 \end{array}\right),
\]

\[
E_z(k,2) = \sum_{i=0}^{2k} \left(\begin{array}{c} 4k+1 \\ 2i+1 \end{array}\right) \left(\begin{array}{c} 4k+1 \\ 4k-2i+1 \end{array}\right),
\]

\[
E_z(k,3) = \sum_{i=0}^{2k+1} \left(\begin{array}{c} 4k+2 \\ 2i+1 \end{array}\right) \left(\begin{array}{c} 4k+2 \\ 4k-2i+2 \end{array}\right).
\]

**Proof.** Let

\[
\sum_{i=1}^{n} q_i = Q(q) \quad \text{and} \quad \sum_{i=1}^{n} q'_i = Q'(q).
\]

According to the value of \( j \) we consider four cases. Since \( q \) is an even sequence, therefore \( Q(q) \) is always even.

1. If \( j = 0 \), then

\[
Q'(q) = Q(q) + \sum_{i=0}^{4k-1} i = Q(q) + 2k(4k-1)
\]

is even, therefore the number of odd elements of \( q' \) is also even. The interval \([1, 8k-2]\) contains \( 8k-2 \) elements and among them \( 4k-1 \) even and \( 4k-1 \) odd elements, so for \( q' \) we can choose \( 2i \) odd elements from \( 4k-1 \) candidates and \( 4k-2i \) \((i = 0, \ldots, 2k-1)\) even elements from \( 4k-1 \) candidates, so we get (26).
2. If \( j = 1 \), then

\[
Q'(q) = Q(q) + \sum_{i=0}^{4k} i = Q(q) + 2k(4k + 1),
\]

is even, therefore the number of odd elements of \( q' \) is also even. The interval \([1, 8k]\) contains \( 8k \) elements and among them \( 4k \) odd and \( 4k \) even elements, so for \( q' \) we can choose \( 2i \) odd elements from \( 4k \) candidates and \( 4k - 2i + 1 \) (i.e., \( i = 0, \ldots, 2k \)) even elements from \( 4k - 1 \) candidates, so we get (27).

3. If \( j = 2 \), then

\[
Q'(q) = Q(q) + \sum_{i=0}^{4k+1} i = Q(q) + (2k + 1)(4k + 1)
\]

is odd, therefore the number of odd elements of \( q' \) is also odd. The interval \([1, 8k + 2]\) contains \( 8k + 2 \) elements and among them \( 4k + 1 \) even and \( 4k + 1 \) odd elements, so for \( q' \) we can choose \( 2i + 1 \) odd elements from \( 4k + 2 \) candidates and \( 4k - 2i - 1 \) (i.e., \( i = 0, \ldots, 2k - 1 \)) even elements from \( 4k + 1 \) candidates, so we get (28).

4. If \( j = 3 \), then

\[
Q'(q) = Q(q) + \sum_{i=0}^{4k+2} i = Q(q) + (2k + 1)(4k + 3),
\]

and so \( Q(q') \) is also odd, therefore \( q' \) contains an odd number of odd elements. The interval \([1, 8k + 4]\) contains \( 8k + 4 \) elements and among them \( 4k + 2 \) even and \( 4k + 2 \) odd elements, so for \( q' \) we can choose \( 2i + 1 \) odd elements from \( 4k + 2 \) candidates and \( 4k - 2i - 1 \) (i.e., \( i = 0, \ldots, 2k - 1 \)) even elements from \( 4k + 2 \) candidates, so

\[
E_z(k, 3) = \sum_{i=0}^{2k+1} \binom{4k + 2}{2i + 1} \binom{4k + 2}{4k - 2i + 2}.
\]

\( \Box \)

Table 1 shows the values of \( R(n)/R(n + 1) \), \( R_z(n)/R_z(n + 1) \), \( E(n)/R(n) \), \( E(n)/E(n + 1) \), \( E_z(n)/E_z(n + 1) \), and \( E_z(n)/R_z(n) \) for \( n = 1, \ldots, 32 \).
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For example let \( n = 1 \), then
\[
\begin{align*}
\frac{R(1)}{E(1)} &= \frac{3}{0} \cdot \frac{3}{4} + \frac{3}{2} \cdot \frac{3}{2} = 1 \cdot 0 + 3 \cdot 3 = 9. \\
\end{align*}
\]

If \( n = 2 \), then \( k = 1, j = 1 \) and
\[
\begin{align*}
E_2(2) &= E_2(1, 0) = \binom{3}{0} \binom{3}{4} + \binom{3}{2} \binom{3}{2} = 1 \cdot 0 + 3 \cdot 3 = 9. \\
\end{align*}
\]

Table 1: The values of \( R(n)/R(n+1) \), \( R_2(n)/R_2(n+1) \), \( E(n)/R(n) \), \( E(n)/E(n+1) \), \( E_2(n)/E_2(n+1) \), and \( E_2(n)/R_2(n) \) for \( n = 1, \ldots, 32 \)

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</table>

It is remarkable that in \( R(101)/R(102) \) and \( R_2(101)/R_2(102) \) the first nine decimal digits are equal.

For example let \( n = 4 \), then \( k = 1, j = 0 \) and
\[
E_2(4) = E_2(1, 0) = \binom{3}{0} \binom{3}{4} + \binom{3}{2} \binom{3}{2} = 1 \cdot 0 + 3 \cdot 3 = 9. \\
\]

If \( n = 5 \), then \( k = 1, j = 1 \) and
\[
E_2(5) = \binom{4}{0} \binom{4}{5} + \binom{4}{2} \binom{4}{3} + \binom{4}{4} \binom{4}{1} = 0 + 24 + 4 = 28. \\
\]
If \( n = 6 \), then \( k = 1 \), \( j = 2 \) and
\[
E_z(6) = \binom{5}{1} \binom{5}{5} + \binom{5}{3} \binom{5}{3} + \binom{5}{5} \binom{5}{1} = 5 + 100 + 5 = 110. \quad (38)
\]
If \( n = 7 \), then \( k = 1 \), \( j = 3 \) and
\[
E_z(7) = \binom{6}{1} \binom{6}{6} + \binom{6}{3} \binom{6}{4} + \binom{6}{5} \binom{6}{2} = 6 + 300 + 90 = 396. \quad (39)
\]
If \( n = 8 \), then \( k = 2 \), \( j = 0 \) and
\[
E_z(8) = \binom{7}{0} \binom{7}{8} + \binom{7}{2} \binom{7}{6} + \binom{7}{4} \binom{7}{4} + \binom{7}{6} \binom{7}{2} = 1519. \quad (40)
\]
Simulation results in Table 1 show, that if \( 1 \leq n \leq 32 \) and \( n \) is odd, then \( E_z(n)/R_z(n) = 0.5 \). This property is true for larger odd \( n \)’s too.

**Lemma 8** If \( 1 \leq k \leq 600 \), then
\[
\frac{E_z(2k - 1)}{R_z(2k - 1)} = 0.5. \quad (41)
\]

**Proof.** See the computed values of \( R_z(n) \) and \( E_z(n) \) in [68].

Table 2 contains the ratios \( E_z(n)/G_z(n) \) for \( n = 23, \ldots, 29 \) and the ratios \( T(n)/G_z(n) \) for \( n = 30 \) and \( n = 31 \).

The data in Table 2 show that the function \( G_z/E_z \) is decreasing. We suppose that this function tends monotonically decreasing to zero when \( n \) tends to infinity (in a similar way as the function \( G(n)/E(n) \) tends to zero according to Corollary 23 [52, page 260].

Table 3 contains the values of \( G_z(n) \), \( T(n) \), and \( G_z(n)/T(n) \) for \( n = 30 \) and \( n = 31 \): the ratio of the graphical and tested sequences is much higher and these ratios are increasing. These changes are due to the fact that EGE2 jumps many nongraphical zero-free even sequences without testing them.

### 3.2 New algorithmic results

Using the following Lemma 9 later we will further fasten EGE.

If \( b = (b_1, \ldots, b_n) \) is a regular sequence, then \( c = (c_1, \ldots, c_n) \) is called **lexicographically i-smaller, than** \( b \) if there exist indices \( i \) and \( j \) such that
\[
1 \leq i < j \leq n,
\]
\[
1 \leq i < j \leq n, \quad (42)
\]
Table 2: The values of $G_z(n)$, $E_z(n)$, and $G_z(n)/E_z(n)$ for $n = 17, \ldots, 31$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$G_z(n)$</th>
<th>$E_z(n)$</th>
<th>$G_z(n)/E_z(n)$</th>
</tr>
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</tr>
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<tr>
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<td>22,974,847,474,172,100</td>
<td>57,224,797,531,384,560</td>
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</table>

Table 3: The values of $G_z(n)$, $T(n)$, and $G_z(n)/T(n)$ for $n = 30$ and $n = 31$

<table>
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<tr>
<th>$n$</th>
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<th>$T(n)$</th>
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</table>

Further

$$c_k = b_k \quad \text{for} \quad k = 1, \ldots, i,$$  \hspace{1cm} (43)

and

$$\sum_{k=i+1}^{n} c_k \leq \sum_{k=i+1}^{n} b_k.$$  \hspace{1cm} (44)

Lemma 9 If $b = (b_1, \ldots, b_n)$ and is a nongraphical sequence and $c = (c_1, \ldots, c_n)$ is lexicographically $i$-smaller than $b$ for some $i$ ($1 \leq i < n$), then $c$ is also graphical.

Proof. See [54].

Using this lemma the running time of EGLJ decreased substantially. It was very useful when we enumerated the edge sequences of the simple graphs on
30 vertices between June 21 and 18 July of 2013 and on 31 vertices between 18 July and 24 August (the results see in Table 4).

Using the results of Tripathi and Vijay [52, Lemma 6 and Theorem 7] we can substantially decrease the average testing time of the zero-free even sequences. It is known that the expected number of checking points proposed by Tripathi and Vijay is about $n/2$ [52].

Algorithm EGE2 [51, pages 274–277] used in the enumerations for $n = 30$ and $n = 31$ vertices is based on Lemma 9.

We develop algorithm EGE3 for the enumeration of the degree sequences in the case of 32 vertices. EGE3 will be based on Lemmas 10 and 11.

**Lemma 10** If the investigated by EG3 sequence is graphical and has the form $b = b_1^{c_1} b_2^{c_1} b_3^{c_2}$ and the upper bound $c_1(c_1 - 1)$ of the inner capacity of the $c_1$-length head of $b$ covers $H_n$, that is if

$$c_1(c_1 - 1) \geq H_n \quad (45)$$

then all sequences starting with $b_1^{c_1}$ are graphical.

**Proof.** See [54]. □

The next lemma allows to enumerate many graphical sequences without their time consuming testing.

**Lemma 11** If the investigated by EG3 graphical sequence has the form $b = b_1^{c_1} b_2^{c_2} \ldots b_p^{c_p} 1^{c_{p+1}}$, where $p \geq 1$, $b_1 > b_2 > \cdots > b_p \geq 3$, $c_1, \ldots, c_{p+1} \geq 1$, then all zero-free even sequences starting with the prefix

$$b_1^{c_1} b_2^{c_2} \ldots b_{p-1}^{c_{p-1}} b_p^{c_p-1} (b_p - 1)$$

are also graphical.

**Proof.** See [54]. □

### 3.3 New simulation results

Table 4 contains the values of $G_z(n)$ and $G(n)$ for $n = 1, \ldots, 31$. The values for $n = 1, \ldots, 9$ were computed by E. Weisstein, for $n = 10, \ldots, 20$ by G. Royle in 2006, for $n = 21, 22$ and $n = 23$ by F. Ruskey in 2006, for $n = 24, \ldots, 29$ by T. Matuszka in January of 2013, for $n = 30$ by L. Lucz in July of 2013 and for $n = 31$ also by L. Lucz in September of 2013 [48, 50, 51, 52, 79].

Column 4 of Table 4 supports the following conjecture formulated by Gordon Royle in 2012.
Conjecture 12 (Royle, 2012). If \( n \) tends to infinity, then \( G_z(n+1)/G_z(n) \) tends to 4.

We think, that the following conjecture is also true.

**Conjecture 13** If \( n \) tends to infinity, then \( G(n+1)/G(n) \) tends to 4.

We observed that when we enumerated these sequences, that in the case \( n = 30 \) vertices 85.40 percent, while in the case \( n = 31 \) vertices 86.67 percent of the investigated potential degree sequences was graphical. Therefore it is useful if we know without a linear time testing that a given tested sequence is graphical.

Figure 1 shows the number of the tested and the graphical sequences as the function of the index of the slices when \( n = 30 \).

Figure 1: The number of tested (trimmed even) sequences and the number of graphical sequences as the function of the index of the slices when \( n = 30 \)

Figure 2 shows the similar data for \( n = 31 \).

We remark that on the site of the journal the Figures 1 and 2 are color (the graphical sequences are represented by red, while the tested sequences by blue color).

Table 5 contains the data of PC’s used for the enumeration of \( G_z(31) \), where Comp. alg. = Computer Algebra, Prog. lang. = Program languages, Core = Core(TM)Kása 1 = Z. Kása (Cluj), Kása 2 = Z. Kása (Tg.-Mureș), Kása 3 =
Table 4: The number $G_z(n)$ of zerofree graphical sequences and the number $G(n)$ of graphical sequences for $n = 1, \ldots, 31$, further the ratios $G_z(n)/G_z(n+1)$ and $G(n)/G(n+1)$ for $n = 1, \ldots, 30$.

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</table>

Z. Kása // (Tg.-Mureș), Sp1 = Speed of a machine in GHz, Sp2 = Speed of the laboratory in GFLOPS, Intel (R) = Intel (R) Xeon (R).

The total number of machines was 350.
Table 6 contains the algorithms, running times and number of jobs in the case $n = 25, \ldots, 31$.

### 3.4 The growth of the functions $R(n), E(n), R_z(n), E_z(n), G(n)$ and $G_z(n)$

In this subsection we present concrete values of the functions characterizing the sizes of the investigated sets of sequences.

The number $R(n)$ of the regular sequences is presented in Figure 1 of [52] for $n = 1, \ldots, 38$ and up to $n = 1200$ in [47].

The values of the zerofree regular $R_z(n)$ can be quickly computed using formula (22) in [47]. The values for $n = 1, \ldots, 1200$ can be found in [68].

The number $E(n)$ of even sequences is presented in Figure 1 of [52] for $n = 1, \ldots, 1000$ in [49] and up to $n = 1200$ in [68].

The number $E_z(n)$ of the zerofree even sequences is contained in Figure 3 of [52] for $n = 1, \ldots, 20$ (these data are the results of brute force simulation) and up to $n = 1200$ in [68].

The order of growth of these functions is $\Theta(4^n/n)$.

According to theorem of Burns [22, 52] the order of growth of $G(n)$ is smaller (see 12).

The known values of $G(n)$ and $G_z(n)$ are summarized in Table 4.
Table 5: Names of laboratories, number of machines, type of machines, speed of machines in GHz, speed of laboratories in GLOPS, used in the case $n = 31$.

3.5 Further plans

Our new program (EGE3) is able to jump the test of some part of zero-free graphical sequences [54]. Due to this property of the new program EGE3 the number of tested sequences is smaller than the number of zero-free graphical ones (see Table 7).

4 Summary

The log files and source codes of our programs can be found at

<table>
<thead>
<tr>
<th>n</th>
<th>Algorithm</th>
<th>Running time (in days)</th>
<th>Running time (in years)</th>
<th>Number of jobs</th>
</tr>
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<tbody>
<tr>
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<tr>
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</tr>
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<td>EGE2</td>
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<td>19.7835</td>
<td>351 155</td>
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<td>89.5954</td>
<td>448 957</td>
</tr>
</tbody>
</table>

Table 6: Number of vertices, used algorithm, total running time (in days and in years) and number of jobs

<table>
<thead>
<tr>
<th>n</th>
<th>$T(n)$</th>
<th>$G_{z}(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
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</tr>
<tr>
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</tbody>
</table>

Table 7: Number of vertices ($n$), number of tested sequences ($T(n)$) and number of zerofree graphical sequences ($G_{z}(n)$)
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