On vertex independence number of uniform hypergraphs

Tariq A. CHISHTI  
University of Kashmir  
Department of Mathematics  
Srinagar, India  
email: chishtita@yahoo.co.in

Guofei ZHOU  
Nanjing University  
Department of Mathematics  
Nanjing, China  
email: gfzhou@mail.nju.edu.cn

Shariefuddin PIRZADA  
University of Kashmir  
Department of Mathematics  
Srinagar, India  
email: pizadasd@kashmiruniversity.ac.in

Antal IVÁNYI  
Eötvös Loránd University  
Faculty of Informatics  
Budapest, Hungary  
email: tony@inf.elte.hu

Abstract. Let $H$ be an $r$-uniform hypergraph with $r \geq 2$ and let $\alpha(H)$ be its vertex independence number. In the paper bounds of $\alpha(H)$ are given for different uniform hypergraphs: if $H$ has no isolated vertex, then in terms of the degrees, and for triangle-free linear $H$ in terms of the order and average degree.

1 Introduction to independence in graphs

Let $n$ be a positive integer. A graph $G$ on vertex set $V = \{v_1, v_2, \ldots, v_n\}$ is a pair $(V, E)$, where the edge set $E$ is a subset of $V \times V$. $n$ is the order of $G$ and $|E|$ is the size of $G$. 

Computing Classification System 1998: G.2.2  
Mathematics Subject Classification 2010: 05C30, 05C50  
Key words and phrases: uniform hypergraph, independence number, lower bound  
DOI:10.2478/ausi-2014-0022
Let $v \in V$ and $N(v)$ be the \textit{neighborhood} of $v$, namely, the set of vertices $x$ so that there is an edge which contains both $v$ and $x$. Let $U$ be a subset of $V$, then the \textit{subgraph} of $G$ induced by $U$ is defined as a graph on vertex set $U$ and edge set $E_U = \{(u,v) | u \in U \text{ and } v \in U\}$.

The \textit{degree} $d(v)$ of a vertex $v \in V$ is the number of edges that contains $v$. Let $d(G)$ be the \textit{average degree} of $G$, then $nd(G) = \sum_{v \in V} d(v) = 2|E|$ for any graph $G$. Let $\delta(G)$ be the \textit{minimal degree}, $\Delta(G)$ the \textit{maximal degree} of $G$. A graph $G$ is \textit{regular} if $\Delta(G) = \delta(G)$, and it is \textit{semi-regular}, if $\Delta(G) - \delta(G) = 1$.

Three vertices $v_1, v_2, v_3$ form a \textit{triangle} in $G$ if there are distinct vertices $e_1, e_2, e_3 \in E$ such that $\{v_i, v_{i+1}\} \subseteq E$, where the indices are taken mod 3. If $G$ does not contain a triangle, then it is \textit{trianglefree}.

A subset $U \subseteq V$ of vertices in a graph $G$ is called a \textit{vertex independent set} if no two vertices in $U$ are adjacent. The maximum-size vertex independent set is called \textit{maximum vertex independent set}. The size of the maximum vertex independent set is called \textit{vertex independence number} and is denoted by $\alpha(G)$. The problem of finding a vertex maximum independent set and vertex independence number are NP-hard optimization problems [73, 167].

A \textit{maximal vertex independent set} is a vertex independent set such that adding any other vertex to the set forces the set to contain an edge. The problem of finding a maximal vertex independent set can be solved in polynomial time (see e.g. the algorithms due to Tarjan and Trojanowski [155], Karp and Widgerson [101], further the improved algorithms due to Luby [128] and Alon [9].

There are exponential time exact (as Alon [9]) and polynomial time approximate algorithms (as Boppana and Haldórsson [30], Agnarsson, Haldórsson, and Losievskaja [4, 5], Losievskaja [126]) determining $\alpha(G)$. Also there are known algorithms producing the list of all maximum independent sets of graphs (see e.g. Johnson and Yannakakis [93], Lawler, Lenstra, Rinnooy Kan [121]).

An \textit{independent edge set} of a graph $G$ is a subset of the edges such that no two edges in the subset share a vertex of $G$ [166]. An independent edge set of maximum size is called a \textit{maximum independent edge set}, and an independent edge set that cannot be expanded to another independent edge set by addition of any other edge in the graph is called a \textit{maximal independent edge set}. The size of the largest independent edge set (i.e., of any maximum independent edge set) in a graph is known as its \textit{edge independence number} (or \textit{matching number}), and is denoted by $\nu(G)$. The determination of $\nu(G)$ is an easy task for bipartite graphs [49, 50], but it is a polynomially solvable problem for general graphs too [10, 101, 161, 162].

Let $G = (V, E)$ be an $n$-order graph. The classical Turán theorem [159] gives
a simple lower bound for $\alpha(G)$.

**Theorem 1** (Turán [159]) If $n \geq 1$ and $G$ is an $n$-order graph, then

$$\alpha(G) \geq \frac{n}{d(G) + 1}. \quad (1)$$

This result was strengthened independently in 1979 by Caro and in 1981 by Wei.

**Theorem 2** (Caro [36], Wei, [165]) If $G(V,E)$ is a graph, then

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v) + 1}. \quad (2)$$

**Proof.** See [36, 165]. □

A nice probabilistic proof of the result can be found in the paper of Alon and Spencer [11]. Since the function $\frac{1}{x+1}$ is convex, $\sum_{v \in V} \frac{1}{d(v) + 1} \geq \frac{n}{d + 1}$. [170].

Since this bound is the best-possible only for graphs which are unions of cliques, additional structural assumptions excluding these graphs allow improvement of 2 [80, 81]. A natural candidate for such assumptions is connectivity. In 2013 Angel, Campigotto, and Laforest [14] improved (2) for some connected graphs. For locally sparse graphs Ajtai, Erdős, Komlós and Szemerédi improved Turán’s bound greatly.

**Theorem 3** (Ajtai, Erdős, Komlós and Szemerédi [6, 7, 8]) If $G$ is an $n$-order triangle-free graph with average degree $d$, then

$$\alpha(G) \geq c n \ln \frac{d}{d + 1}. \quad (3)$$

**Proof.** See [6, 7, 8]. □

They conjectured that $c = 1-o(1)$ when $d$ tends to $\infty$. Griggs [72] improved that $c$ can be $\frac{5}{12}$. Shearer [152] finally proved $c = 1 - o(1)$, thus confirming the conjecture. In 1994 Selkow improved the bound due to Caro and Wei supposing that the degrees of the neighbors of the vertices are also known.

**Theorem 4** (Selkow [150]) If $G(V,E)$ is a graph, then

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v) + 1} \left( 1 + \max \left( 0, \frac{d(v)}{d(v) + 1} - \sum_{u \in N(v)} \frac{1}{d(u) + 1} \right) \right). \quad (4)$$
On vertex independence number of uniform hypergraphs

Proof. See [150]. □

The bound of Selkow is equal to Caro–Wei bound for regular graph and always less than twice the Caro–Wei bound. A recent review on lower bounds for 3-order graphs was published by Henning and Yeo [89].

Let \( j \) and \( k \) be a positive integers. A subset \( I \subseteq V(G) \) is a vertex-k-independent set of \( G \), if every vertex in \( I \) has at most \( k - 1 \) neighbors in \( I \). The vertex-k-independence number \( \alpha_k(G) \) of \( G \) is the cardinality of the largest vertex-k-independent set of \( G \).

A subset \( D \subseteq V(G) \) is a vertex-j-dominating set of \( G \), if every vertex of \( D \) has at least \( j - 1 \) neighbors in \( D \). The vertex-j-independence number \( \gamma_j(G) \) of \( G \) is the cardinality of the largest vertex-j-dominating set of \( G \).


Last year Hansberg and Pepper [79] investigated the connection between \( \alpha_k(G) \) and \( \gamma_j(G) \). They proved the following theorems.

**Theorem 5** (Hansberg, Pepper [79]) If \( G \) be an \( n \)-order graph, \( j \), \( k \) and \( m \) be positive integers such that \( m = j + k - 1 \) and let \( H_m \) and \( G_m \) denote, respectively, the subgraphs induced by the vertices of degree at least \( m \) and the vertices of degree at least \( m \). Then

\[
\alpha_k(H_m) + \gamma_j(G_m) \leq n
\]

and

\[
\alpha_k(G) + \gamma_j(G) = n(G_m).
\]

Proof. See [79]. □

**Theorem 6** (Hansberg, Pepper [79]) Let \( G \) be a connected \( n \)-order graph with maximum degree \( \Delta \) and minimum degree \( \delta \geq 1 \). Then

\[
\alpha_k(G) + \gamma_j(G) = n(G) \quad \text{and} \quad \alpha_k'(G) + \gamma_j'(G) = n(G)
\]
for every pair of integers \( j, k \) and \( j', k' \) such that \( j+k-1 = \delta \) and \( j'+k'-1 = \Delta \) if and only if \( G \) is regular.

**Proof.** See [79]. \( \square \)

**Theorem 7** (Hansberg, Pepper [79]) For any graph \( G \) the following two statements are equivalent:

\[
\gamma(G) + \alpha_s(G) = \kappa(G) \tag{8}
\]

and

\[
G \text{ is regular or } \gamma(G) + \gamma_2(G) = \kappa(G). \tag{9}
\]

**Proof.** See [79]. \( \square \)

Spencer [153] also published some extension of Turán theorem.

In 2014 Henning, Löwenstein, Southey and Yeo [87] proved the following theorem, which is an improvement of the result due to Fajtlowicz [53].

**Theorem 8** (Henning et al. [87]) If \( G \) is a graph of order \( n \) and \( p \) is an integer, such that for every clique \( X \) in \( G \) there exists a vertex \( x \in X \) such that \( d(x) < p - |X| \), then \( \alpha(G) \geq 2n/p \).

There are results on the independence number of random graphs (e.g. Balogh, Morris, Samotij [18] and Frieze [60], Henning, Löwenstein, Southey and Yeo [87], on the weighted independence number (see e.g. Halldórsson [75], Kako, Ono, Hirata, and Halldórsson [98], further Sakai, Mitsunori, and Yamazaki [149]), and on the enumeration of maximum independent sets (see e.g. Gaspers, Kratsch, and Liedloff [69].

Let \( G(n,p) = (V,E) \) the random graph with vertex set \( V = \{v_1, \ldots, v_n\} \), \( p, \alpha(G_n,p) \) denote the independence number of \( G_{n,p} \). In 1990 Frieze [60] proved, that if \( d = np \) and \( \epsilon > 0 \) is fixed, then with probability going to 1 as \( n \to \infty \)

\[
\left| \alpha(G_n,p) - \frac{2n(\ln d - \ln \ln d - \ln 2 + 1)}{d} \right| \leq \frac{\epsilon n}{d}, \tag{10}
\]

provided \( d_\epsilon \leq d = o(n) \), where \( d_\epsilon \) is some fixed constant and \( p \) is the join probability for each edge to be included in \( E \).

In 1983 Shearer proved the following lower bound.

**Theorem 9** (Shearer [152]) If \( G \) is triangle-free, then

\[
\alpha(G) \geq n f(d), \tag{11}
\]
where
\begin{equation}
    f(x) = \frac{x \ln x - x + 1}{(x - 1)^2},
\end{equation}
\begin{equation}
    f(0) = 1 \text{ and } f(1) = \frac{1}{2}.
\end{equation}

According to the proof of Shearer for $0 < x < \infty$ hold $0 < f(d) < 1$, $f'(d) < 0$ and $f''(d) < 0$. Further $f(x)$ satisfies the differential equation
\begin{equation}
    (x + 1)f(x) = (x + 1)d^2f'(x).
\end{equation}

It is easy to see that
\begin{equation}
    \lim_{x \to \infty} \frac{f(x)}{x} = \frac{\ln x}{x}.
\end{equation}

In 1995 Füredi [62] determined the number of different vertex maximal independent set in path graphs.

It is known [22] a minimum covering set of $G$ is also a maximum vertex independent set of $G$. Therefore we are interested in the results on dominating sets (see e.g. [41, 54, 79, 82, 143].

The structure of the paper is as follows. After this introduction in Section 2 we present a review of results connected with th vertex and edge independence number of hypergraphs, then in Section 3 a lower bound of $\alpha(H)$ is presented for $n$-order $r$-uniform hipergraphs with average degree $d(H)$, and finally in Section 4 a similar bound is proved for hypergraphs not containing isolated vertex.

2 Introduction to independence in hypergraphs

Let $n \geq 1$ and $W = \{w_1, w_2, \ldots, w_n\}$ be a finite set called vertex set. A hypergraph $H$ on vertex set $W$ is a pair $(W, F)$, where the edge set $F$ is a family of the elements of $W$. We always assume that distinct edges are distinct as subsets. If each edge in $F$ contains exactly $r \geq 2$ vertices, then $H$ is a $r$-uniform hypergraph. So any graph $G$ is a 2-uniform hypergraph.

Let $w \in W$ and $N(w)$ be the neighborhood of $w$, namely, the set of vertices $x$ so that there is an edge which contains both $w$ and $x$. Let $U$ be a subset of $W$. The sub-hypergraph of $H$ induced by $U$ is defined as a hypergraph on vertex set $U$ with edge set $F_U = \{f \in F : f \subseteq U\}$.

The degree $d(w)$ of a vertex $w \in W$ is the number of edges that contain $w$. Let $d(H) = d$ be the average degree of an $r$-uniform $H$, then $nd = \sum_{w \in W} d(w) = r|F|$. 

For the simplicity we usually omit $G$ and $H$ as arguments of $d(H)$ and similar notations.

A hypergraph $H$ is linear, if any two edges of $H$ have at most one vertex in common. Note that a graph $G$ is always linear. Three vertices $w_1, w_2, w_3$ form a triangle in $H$, if there are distinct edges $f_1, f_2, f_3 \in F$ such that $\{f_i, f_{i+1}\} \subseteq F$, where the indices are taken mod 3.

A subset $U \subseteq W$ of vertices in a hypergraph $H$ is called a vertex independent set if no two vertices in $U$ are adjacent. The maximum-size vertex independent set of $H$ is called maximum vertex independent set. The size of the maximum vertex independent set is called vertex independence number and is denoted by $\alpha(H)$. The problem of finding a maximum vertex-independent set and vertex independence number are NP-hard optimization problems [73, 167].

There are exponential time exact (as Alon [9], Tarjan and Trojanowski [155]) and polynomial time approximate algorithms (as Boppana and Hådorsson [30], Agnarsson, Hådorsson, and Losievskaja [4, 5], Losievskaja [126]). Also there are known algorithms producing the list of all maximum independent sets of graphs (see e.g. Johnson and Yannakakis [93], Lawler, Lenstra, Rinnooy Kan [121]) and hypergraphs (see e.g. Kelsen [107]).

A maximal vertex independent set is a vertex independent set such that adding any other vertex to the set forces the set to contain an edge. The problem of finding a maximal vertex independent set can be solved in polynomial time (see e.g. the algorithms due to Tarjan and Trojanowski [155], Karp and Widgerson [101], further the improved algorithms due to Luby [128] and Noga [9]).

In 2012 Dutta, Mubayi, and Subramanian [48] gave new lower bond for the vertex independence number of sparse hypergraphs.

In 2013 Eustis devoted a PhD dissertation to the problems of hypergraph independence numbers [51, 52].

An independent edge set of a hypergraph $H$ is a subset of the edges such that no two edges in the subset share a vertex of $H$ [136]. An independent edge set of maximum size is called a maximum independent edge set, and an independent edge set that cannot be expanded to another independent edge set by addition of any other edge in the hypergraph is called a maximal independent edge set. The size of the largest independent edge set (i.e., of any maximum independent edge set) in a hypergraph is known as its edge independence number (or matching number), and is denoted by $\nu(H)$. The determination of $\nu(H)$ is an easy task for bipartite graphs [49, 50], but it is a polynomially solvable problem for general graphs too [10].

There are many results on the characterization of hypergraph score se-
quences and on their reconstruction (see e.g. [20, 110, 140, 171, 139, 164, 172]), on the enumeration of different hypergraphs (see e.g. [21, 47, 138, 144, 145]) and directed hypergraphs (see e.g. [15]).

An \( r \)-uniform hypergraph with \( n \) vertices is called \textit{complete}, if its set of edges has the cardinality \( \binom{n}{r} \). The \textit{complement} of an \( r \)-uniform hypergraph \( H \) is \( \overline{H} = (W, \overline{F}) \), if \( |F \cup \overline{F}| = \binom{n}{2} \) and \( |F \cap \overline{F}| = 0 \).

A set \( P \subseteq W \) is called an \textit{edge cover} of \( H \), if for any non-isolated vertex \( x \in W \) there exists an edge \( f_i \in P \) that \( x \in f_i \). The cardinality of a minimum set which is an edge covering of \( H \) is called the \textit{edge covering number} of \( H \), and is denoted by \( \nu(H) \).

The following lemma, proved in [97], gives a relation between the edge covering number and the edge independence number in an \( r \)-uniform hypergraph \( H \) without isolated vertices.

\textbf{Lemma 10 (Jucovič, Olejník [97])} For an \( r \)-uniform \( n \)-order hypergraph \( H \) with \( n \) without isolated vertices the following inequalities hold:

\[
\alpha(H) \leq n - (kr - 1)\nu(H), \tag{15}
\]

\[
\alpha(H) + (r - 1)\nu(H) \leq n. \tag{16}
\]

\[
\nu(H) + (r - 1)r - 1\nu(H) \geq n, \tag{17}
\]

\textbf{Proof.} See [97].


In 1989 Olejník proved the following three theorems characterizing \( \alpha(H) \) and \( \nu(H) \).

\textbf{Theorem 11 (Olejník [136])} For an \( r \)-uniform \( n \)-order hypergraph \( H = (W, F) \) with \( n \) and its complement \( \overline{H} = (W, \overline{F}) \)

\[
\left\lfloor \frac{n}{r} \right\rfloor \leq \nu(H) + \nu(\overline{H}) \leq 2 \left\lfloor \frac{n}{r} \right\rfloor. \tag{18}
\]

\text{and}

\[
0 \leq \nu(H)\nu(\overline{H}) \leq \left\lfloor \frac{n}{r} \right\rfloor^2. \tag{19}
\]

\textbf{Proof.} See [136].

This bounds are direct generalizations of the bounds published by Chartrand and Schuster in 1974 [40].
Theorem 12 (Olejník [136]) For an $r$-uniform $n$-order hypergraph $H = (W, F)$ and its complement $\overline{H} = (W, \overline{F})$, where neither $H$ nor $\overline{H}$ have isolated vertices,

$$\left\lfloor \frac{n}{r} \right\rfloor \leq \nu(H) + \nu(\overline{H}) \leq 2 \left\lfloor \frac{n}{r} \right\rfloor$$

(20)

and

$$0 \leq \nu(H) \nu(\overline{H}) \leq \left\lfloor \frac{n}{r} \right\rfloor^2.$$

(21)

Proof. See [136]. □

This result is an extension of the work of R. Laskar and B. Auerbach published in 1978 [120].

Theorem 13 (Olejník [136]) For an $r$-uniform $n$-order hypergraph $H = (W, F)$ and its complement $\overline{H}, \overline{F}$, where neither $H$ nor $\overline{H}$ have isolated vertices and $n \neq 2r$

$$2 \left\lfloor \frac{n}{r} \right\rfloor \leq \alpha H + \alpha \overline{H} \leq 2n - (r - 1) \left\lfloor \frac{n}{r} \right\rfloor - r + 1$$

(22)

and

$$\left\lfloor \frac{n}{r} \right\rfloor^2 \leq \alpha(H) \alpha(\overline{H}) \leq \frac{1}{4} \left(2n - (r - 1) \left\lfloor \frac{n}{r} \right\rfloor - k + 1\right)^2.$$

(23)

Proof. See [136]. □


Let

$$B(p, q) = \int_0^1 (1 - t)^{p-1} t^{q-1} dt$$

(24)

denote the beta-function with $p, q > 0$. Set constants $0 < a \leq 1, 0 < b \leq 1$, and $B = B(a, 1 - b)$, and let

$$f_r(x) = \frac{1}{B} \int_0^1 \frac{1 - t}{{t^b[1 + (x - 1)t]}^a} dt.$$

(25)

In 2004 Zhou and Li [170] proved the following theorem on sparse hypergraphs.

Theorem 14 (Zhou, Li [170]) Let $H$ be a triangle-free, $r$-uniform ($r \geq 2$) $n$-order linear hypergraph with average degree $d$. Then its strong vertex independence number $\alpha_s(G)$ is at least $nf_r(d)$.
Proof. See [170]. □


Shearer’s result ([152], further (11) and (12)) was generalized in [170] with the function \( g_r(x) \) satisfying

\[
(r - 1)^2 x(x - 1)g_r'(x) + [(r - 1)x + 1]g_r(x) = 1 \quad (26)
\]

for \( r \)-uniform, triangle-free linear hypergraphs, with sparse neighborhood and in [125] with the function \( g_{r,m}(x) \) satisfying

\[
(r - 1)^2 x(x - m)g_{r,m}'(x) + [(r - 1)x + 1]g_{r,m}(x) = 1 \quad (27)
\]

for \( r \)-uniform, triangle-free, and double linear hypergraphs, in which each subhypergraph induced by a neighborhood, has maximum degree less than \( m \).

A linear hypergraph is called double linear if for any non-adjacent distinct vertices \( w \) and \( z \), each edge containing \( w \) has at most one neighbor of \( z \). From the uniqueness of solutions of the differential equations, we see that \( g_2(x) = g(x) \) and \( g_{r,1}(x) = g_r(x) \). It is shown [125] that \( g_{2,m}(x) \sim \frac{\log x}{x} \), and for \( g_{r,m}(x) \sim \frac{c}{d^{1/(r-1)}} \) for \( r \geq 3 \), where \( c = c(r, m) > 0 \) is a constant without knowing exact values.

Independent sets and numbers are studied in many papers (see e.g. the papers of Abraham [1], Alon, Uri and Azar [12], Berger and Ziv [23], Bollobás, Daykin and Erdős [27], Bonato, Brown, Mitsche and Pralat [28, 29], Bordewich, Dyer and Karpiński [31], Boros, Gurvich, Elbassioni, Gurvich and Khachiyan [32, 33], Borowiecki and Michalak [34], Cutler and Radcliffe [45], Greenhill [70], Halldórsson and Losievskaja [76], Hofmeister and Lehman [90], Johnson and Yannakakis [93], Khachiyan, Boros, Gurvich, and Elbassioni [108], Lepin [122], Li and Zhang [125], Losievskaja [126], Shachnai and Srinivasan [151], Tarjan and Trojanowski [155], Yuster [168]).

Since independence number and matching number are closely connected, we are interested in the results on maximum matching algorithms too (see e.g. [25, 26, 46, 47, 49, 50, 56, 57, 61, 65, 66, 77, 78, 86, 88, 89, 91, 92, 100, 104, 105, 109, 112, 113, 118, 119, 127, 131, 132, 133, 135, 137, 142, 146, 147, 148, 154, 157, 158, 169]).
Minimum dominating set of $H$ and maximum vertex independent set of $H$ are connected concepts, therefore we are interested in the results on dominating sets of hypergraphs (see e.g. [2, 96]).

Further connected problems are also often analyzed (see e.g. e.g. in the papers of Agnarsson, Egilsson, and Halldórson [3], Alon, Frankl, Huan, Rödl, Ruciński [10], Alon and Yuster [13], Baranyai [19], Balogh, Butterfield, Hu and Lenz [17], Bertram-Kretchberg and Letzman [24], Bujtás and Tuza [35], Cockayne, Hedetniemi, and Laskar [43], Frank, Király and Király [55], Frankl and Rödl [58, 59], Füredi, Ruszinkó, and Selver [63, 64], Hán, Person and Schacht [78], Henning and Yeo [89], Huang, Loh and Sudakov [92], Johnson and Yannakakis [93], Johnston and Lu [94, 95], Jucovič and Olejník [97], Karonski and Łuczak [99], Katona [102, 103], Keevash and Sudakov [106], Kelsen [107], Kohayakawa, Rödl, Skokan [111], Krivelevich [115], Kühn and Loose [117], Kostochka, Mubayi, Verstraëte [114], Krivelevich, Nathaniel, and Sudakov [116], Li, Rousseau and Zang [123, 124], Łuczak and Szmański [129, 134], Szmański [154], Treglown and Zhao [157, 158], Tuza [160], Yuster [169]).

Although hypergraphs are less often used in the practice than the graphs, they also have different applications in the practice.

For example Bailey, Manoukian, Ramamohanaro [16], further Gunopoulos, Khardon, Mannila and Toivonen [74] reported on the applications in data mining, Gallo, Longo, Nguyen, and Pallottino [68], further and Maier [130] in relational databases.

In 2000 Carr, Lancia, Istrail, and Genomics [39] reported on Branch-and-Cut algorithms for vertex independent set problem and on their application to solve problems connected with protein structure alignment.

In this paper, we obtain $\alpha(H) \geq \sum_{v \in V} \frac{1}{d(v)^{1/r-1}}$ for any $r$-uniform hypergraph $H$ without the condition of being triangle-free. The algorithm is naive: it deletes a vertex of maximum degree repeatedly. In order to get a large independent set, a commonly used algorithm is to find a suitable vertex $v$, then delete $v$ and its neighbors, and then do the iterations. Deleting all neighbors seems to be of no use for hypergraphs as in [125, 170]. After deleting a vertex $v$, we delete only one vertex other than $v$ from each edge containing $v$. Our new function $f_r(x)$ satisfies

$$(r - 1)x^2 - x)f'_r(x) + (x + 1)f_r(x) = 1. \quad (28)$$

Then $f_r(x) \sim \frac{c}{x^{(r-1)}}$ as $x \to \infty$. We do not know the exact value of $c = c(r)$. However, when we run the algorithm, we note that for a vertex $v$, we delete $1 + d(v)$ vertices instead of deleting $1 + (r - 1)d(v)$ vertices as in [125, 170]. So
if $c$ is the constant such that $g_r(x) \sim \frac{c}{x^{1/(r-1)}}$ as $x \to \infty$, then the new constant seems to be $(r-1)c$, namely, $f_r(x) \sim \frac{(r-1)c}{x^{1/(r-1)}}$.

3 Bound for uniform hypergraphs without isolated vertex

The following Theorem 15 is a corollary of Theorem 18, but it has an easy probabilistic proof.

**Theorem 15** Let $H = (V, E)$ be an $r$-uniform hypergraph of order $n$ and average degree $d \geq 1$, then

$$\alpha(H) \geq \left(1 - \frac{1}{r}\right) \frac{n}{d^{1/(r-1)}}. \quad (29)$$

**Proof.** Define a random subset $U \subseteq V$ by $\Pr(v \in U) = p$ for some $0 \leq p \leq 1$ with all these events being mutually independent over $v \in V$.

Let $X(U)$ be the number of vertices in $U$ and let $Y(U)$ be the number of edges in the subgraph induced by $U$. Note that for one of the edges of $H$, the probability that all of its vertices belong to $U$ is $p^r$. By linearity of expectation, we have

$$E(X - Y) = E(X) - E(Y) = np - \frac{nd}{r}p^r. \quad (30)$$

Thus there exists a set $U$ satisfying

$$X(U) - Y(U) \geq E(X) - E(Y). \quad (31)$$

Note that $U$ is not that we require, since the sub-hypergraph of $H$ induced by $U$ may have edges. However, if we delete one vertex from each edge contained in $U$, then at most $Y(U)$ vertices are deleted, we thus obtain a new set with at least $E(X) - E(Y)$ vertices and whose induced sub-hypergraph has no edges. The desired lower bound follows by taking $p = \frac{1}{d^{1/(r-1)}}$. $\square$

For hypergraphs that are not regular, Theorem 18 is stronger than Theorem 15. We need two lemmas for the proof of Theorem 18.

**Lemma 16** Let $r \geq 2$ be an integer and define

$$h_r(x) = \begin{cases} 1 - x/r & \text{if } 0 \leq x < 1 \\ \frac{1}{x^{1/r}} & \text{if } x \geq 1, \end{cases} \quad (32)$$

then $h_r(x)$ is positive, decreasing and convex. Furthermore, for $x \geq 1$, the function $h_r(x)$ satisfies that $(r-1)x h'(x) + h_r(x) = 0$. 


Proof. It is easy to see that \( h_r(x) \) is positive and
\[
h'_r(x) = \begin{cases} 
-1/r & \text{if } 0 \leq x < 1 \\
-x/((r-1)x^{r-1}) & \text{if } x \geq 1.
\end{cases}
\] (33)

So \( h'_r(x) \) is continuous, negative and increasing, thus \( h_r(x) \) is decreasing and convex. The fact that \( h_r(x) \) satisfies the mentioned differential equation is straightforward. \( \square \)

Let \( \Delta = \Delta(H) \) denote the maximal degree in \( H \) and define
\[
S(G) = \sum_{x \in V} h(d(x)), \quad S(H) = \sum_{x \in W} h(d(x)). \quad (34)
\]

Lemma 17 If \( \Delta(H) \geq 1 \), \( w \in W \), \( d(w) = \Delta(H) \), and \( H_1 = H - \{w\} \), then \( S(H_1) \geq S(G) \).

Proof. For each \( x \in V \setminus \{v\} \), denote by \( n_x \) the number of edges of \( H \) that contain both \( x \) and \( v \). Then \( n_x = 0 \) if \( x \) and \( v \) are not adjacent, and \( n_x \geq 1 \) otherwise. It is easy to see
\[
\sum_{x \in V \setminus \{v\}} n_x = (r-1)\Delta \quad (35)
\]
since \( H \) is \( r \)-uniform. On the other hand, we have
\[
S(H_1) = S(H) - h(\Delta) + \sum_{x \in V \setminus \{v\}} [h(d(x) - n_x) - h(d(x))]. \quad (36)
\]
From the fact that \( h'(x) \) is negative and increasing, we have
\[
h(d(x) - n_x) - h(d(x)) = -h'(\theta_x)n_x \geq -h'(\Delta)n_x, \quad (37)
\]
where \( \theta_x \in [d(x) - n_x, d(x)] \), thus
\[
S(H_1) \geq S(H) - h(\Delta) - h'(\Delta) \sum_{x \in V \setminus \{v\}} n_x
= S(H) - h(\Delta) - (r-1)\Delta h'(\Delta)
= S(H),
\]
proving the claim. \( \square \)
Theorem 18 Let $H = (V, E)$ be an $r$-uniform hypergraph without isolated vertex, then
\[ \alpha(H) \geq \left( 1 - \frac{1}{r} \right) \sum_{v \in V} \frac{1}{d(v)^{1/(r-1)}}. \] (38)

Proof. We write $h_r(x)$ as $h(x)$ for simplicity and define
\[ S(H) = \sum_{x \in V} h(d(x)). \] (39)

Repeat the algorithm by deleting the vertex of maximum degree if the degree is at least one, terminate the algorithm if there are no edges. Denote by $H_0 = H, H_1, ..., H_\ell$ for the sequence of hypergraphs, where $H_\ell$ has no edge. We get $S(H_\ell) = n - \ell$ since $h(0) = 1$, where $n - \ell$ is the order of $H_\ell$, and $\alpha(H) \geq n - \ell$. So
\[ \alpha(H) \geq S(H_\ell) \geq S(H_{\ell-1}) \geq \cdots \geq S(H_0) = S(H), \] (40)
the assertion follows immediately. \hfill \square

Since the function $\frac{1}{x^{1/(r-1)}}$ is convex, Theorem 15 is truly a corollary of Theorem 18.

Remark. Theorem 18 gives $\alpha(G) \geq \sum_v \frac{1}{2d(v)}$ for a graph $G$ with $\delta(G) \geq 1$, which is weaker than $\alpha(G) \geq \sum_v \frac{1}{d(v)+1}$. However, the later can be proved similarly by replacing the function $h(x)$ with $1/(x+1)$. For details of this algorithm, see Griggs [72].

4 Bound for uniform linear triangle-free hypergraphs

In this section triangle-free hypergraphs are considered. To generalize Shearer’s method [152] and to delete less vertices for a hypergraph, we have a definition as follows.

Let $H = (V, E)$ be an $r$-uniform hypergraph and let $v$ be a vertex of $H$, denote by $E_v = \{ e \in E : v \in e \} = \{ e_1, e_2, ..., e_{d(v)} \}$ for the set of edges containing $v$. A claw of $v$ is a set of neighbors of $v$ of the form $\{ u_1, u_2, ..., u_{d(v)} \}$ such that each $u_i \in e_i - v$. For a claw $T$ of $v$, we write as $Q_T$, the number of edges that intersect $T$.

When we run the algorithm in each step, we will delete $v$ and a claw $T$, so $Q_T$ edges will be deleted. The new function is as follows.
Let \( r \geq 2 \) be and integer and let \( b = \frac{r^2}{r-1} \). Define
\[
f_r(x) = \frac{1}{r-1} \int_0^1 \frac{1-t}{t^b[1 + ((r-1)x-1)t]} dt. \tag{41}
\]

**Lemma 19** The function \( f_r(x) \) satisfies the differential equation
\[
([r-1]x^2 - x)f_r'(x) + (x+1)f_r(x) = 1, \tag{42}
\]
and it is positive, decreasing and convex.

**Proof.** By differentiating under the integral and then integrating by parts, we have
\[
\begin{align*}
((r-1)x^2 - x)f_r'(x) & = -([r-1]x^2 - x) \int_0^1 \frac{1-t}{t^{1-b}[1 + ((r-1)x-1)t]^2} dt
\end{align*}
\]
\[
\begin{align*}
& = x \int_0^1 (1-t)t^{1-b} \frac{d}{dt} \left( \frac{1}{1 + [(r-1)x-1]t} \right) dt
\end{align*}
\]
\[
\begin{align*}
& = -x \int_0^1 \frac{1}{1 + [(r-1)x-1]t} [(1-t)(1-b)t^{-b} - t^{-b}] dt
\end{align*}
\]
\[
\begin{align*}
& = -(r-1)(1-b)xf_r(x) + x \int_0^1 \frac{t^{1-b}}{1 + [(r-1)x-1]t} dt
\end{align*}
\]
\[
\begin{align*}
& = -xf_r(x) + \frac{1}{r-1} \int_0^1 \left( \frac{1}{1-t} - \frac{1}{1 + [(r-1)x-1]t} \right) (1-t)t^{-b} dt
\end{align*}
\]
\[
\begin{align*}
& = 1 - (x+1)f_r(x)
\end{align*}
\]
which follows by the differential equation. The monotonicity and convexity of \( f_r(x) \) can be seen by repeated differentiation under the integral. \( \square \)

**Theorem 20** Let \( H \) be an \( r \)-uniform \( n \)-order hypergraph with average degree \( d \). If it is triangle-free and linear, then \( \alpha(H) \geq nf_r(d) \).

**Proof.** We apply induction on \(|V|\), the number of vertices of \( H \). The result is trivial for \(|V| = 1\), since \( f(0) = 1 \). Since the case \( r = 2 \) is exactly what Shearer has given, we suppose that \( r \geq 3 \).
On vertex independence number of uniform hypergraphs

For each \( v \in H \), let \( T = \{u_1, u_2, \ldots, u_{d(v)}\} \) be a claw of \( v \). Since \( H \) is \( r \) uniform, linear and triangle-free, we have

\[
Q_T = d(v) + \sum_{i=1}^{d(v)} (d(u_i) - 1) = \sum_{i=1}^{d(v)} d(u_i).
\]

(43)

Let \( T_v \) be the set of all claws of \( v \), then \( |T_v| = (r - 1)^{d(v)} \). Therefore

\[
\sum_{T \in T_v} Q_T = \sum_{T \in T_v} \sum_{i=1}^{d(v)} d(u_i) = \sum_{u \in N(v)} (r - 1)^{d(v) - 1} d(u),
\]

(44)

and

\[
\frac{1}{|T_v|} \sum_{T \in T_v} Q_T = \sum_{u \in N(v)} \frac{d(u)}{r - 1}.
\]

(45)

We write \( f(x) \) for \( f_r(x) \) and set

\[
R_T(v) = 1 - (d(v) + 1)f(d) + (dd(v) + d - rQ_T)f'(d).
\]

(46)

Then the average of \( R_T(v) \) among \( T \in T_v \) is

\[
\frac{1}{|T_v|} \sum_{T \in T_v} R_T(v) = 1 - (d(v) + 1)f(d) + (dd(v) + d + dQ_T)f'(d) - r \sum_{u \in N(v)} \frac{d(u)}{r - 1} f'(d).
\]

(47)

Note that

\[
\frac{1}{n} \sum_{v \in V} \sum_{u \in N(v)} \frac{d(u)}{r - 1} = \frac{1}{n} \sum_{v \in V} d^2(v) \geq d^2
\]

(48)

as \( x^2 \) is a convex function. Since \( f'(x) < 0 \), we have

\[
\frac{1}{n} \sum_{v \in V} \frac{1}{|T_v|} \sum_{T \in T_v} R_T(v) \geq 1 - (d + 1)f(d) + (d^2 + d - rd^2)f'(d) = 0.
\]

(49)

Hence there exists a vertex, say \( v \), and a claw of \( v \), say \( T = \{u_1, u_2, \ldots, u_{d(v)}\} \), such that \( R(v) \geq 0 \). Now by deleting \( v \) and \( u_1, u_2, \ldots, u_{d(v)} \), we obtain a new hypergraph \( H' \) with \( n - d(v) - 1 \) vertices and \( \frac{nd}{r} - Q_T \) edges. For an edge \( e \) containing \( v \), it contains \( r \geq 3 \) vertices, and we delete exactly two vertices from \( e \), so \( H' \) has some vertices. Note that the average degree \( \bar{d} \) of \( H' \) is \( \frac{nd - rQ_T}{n - d(v) - 1} \).

By induction hypothesis, we have

\[
\alpha(H) \geq (n - d(v) - 1)f(\bar{d}) = (n - d(v) - 1)f \left( \frac{nd - rQ_T}{n - d(v) - 1} \right).
\]

(50)
Combining the facts that \( \alpha(H) \geq 1 + \alpha(H') \) and \( f(x) \geq f(d) + f'(d)(x - d) \) for all \( x \geq 0 \) as \( f(x) \) is convex, we obtain
\[
\alpha(H) \geq 1 + (n - d(v) - 1)f \left( \frac{nd - rQ_T}{n - d(v) - 1} \right)
\geq 1 + (n - d(v) - 1)f(d) + (dd + d - rQ_T)f'(d)
= nf(d) + R(v) \geq nf(d)
\]
completing the proof. \( \square \)

We now get an asymptotic form of \( f_r(x) \) as \( c \frac{x}{1/(r-1)} \) without knowing exact expression of \( c = c(r) \) in hope of improving the old constant based on analysis of the algorithm as mentioned.

**Lemma 21** Let \( r \geq 3 \) be an integer. Then
\[
\lim_{x \to \infty} f_r(x) = \frac{c}{x^{1/(r-1)}},
\]
where \( c = c(r) \) is a positive constant.

**Proof.** Recall that a first order linear differential equation \( \frac{dy}{dx} = p(x)y + q(x) \) has the unique solution of the form
\[
y(x) = e^{\phi(x)} \left( y_0 + \int_{x_0}^{x} q(t)e^{-\phi(t)} \, dt \right)
\]
satisfying \( y_0 = y(x_0) \), where \( \phi(x) = \int_{x_0}^{x} p(t) \, dt \). From the differential equation that \( f_r(x) \) satisfies, we set
\[
p(x) = -\frac{x + 1}{(r-1)x^2 - x}, \quad \text{and} \quad q(x) = \frac{1}{(r-1)x^2 - x}.
\]

For \( x_0 = 2 \),
\[
\phi(x) = -\int_{2}^{x} \frac{t + 1}{(r-1)t^2 - t} \, dt = \ln \frac{c_1 x}{[(r-1)x - 1]^{1/(r-1)}}
\]
Hence
\[
e^{\phi(x)} = \frac{c_1 x}{[(r-1)x - 1]^{1/(r-1)}} \sim \frac{c_2}{x^{1/(r-1)}},
\]
Then we have
\[
q(t)e^{-\phi(t)} \sim \frac{1}{c_2(r-1)} x^{1/(r-1)-2},
\]
On vertex independence number of uniform hypergraphs

implying that $c_3 = \int_2^\infty q(t)e^{-\phi(t)}\,dt < \infty$, and $\int_2^\infty q(t)e^{-\phi(t)}\,dt = c_3 + o(1)$ as $x \to \infty$. Therefore,

$$f_r(x) = e^{\phi(x)} (y_0 + c_3 + o(1)) \sim \frac{c}{x^{1/(r-1)}}, \quad (57)$$

where $c = c_2(y_0 + c_3)$ and $y_0 = f_r(2)$ are positive constants. □

**Acknowledgement** The authors thank Attila Kiss (Eötvös Loránd University) and the unknown referee for the useful comments. This research is supported in part by the National Science Foundation of China (No. 11371193).

**References**


On vertex independence number of uniform hypergraphs 151


[38] Y. Caro, Zs. Tuza, Improved lower bounds on k-independence, *J. Graph Theory* 15 (1991) 99–107. ⇒ 135


[46] B. C. Dean, S. M. Hedetniemi, S. T. Hedetniemii, J. Lewis, A. McRae, Match-
number of sparse graphs and hypergraphs. *SIAM J. Discrete Math.* 26, 3
(2012) 1134–1147. ⇒138
⇒133, 138, 141
[50] J. Edmonds, Maximal matching and a polyhedron with 0, 1-vertices, *J. Research
[52] A. Eustis, J. Verstraëte, On the independence number of Steiner systems, *Com-
[53] S. Fajtlowicz, Independence, clique size and maximum degree, *Combinatorica* 4
(1984), 35–38. ⇒136
[54] O. Favoron, A. Hansberg, L. Volkmann, On k-domination and minimum degree
[55] A. Frank, T. Király, Z. Király, On the orientation of graphs and hypergraphs,
J. Combin.* 19, 2 (2012), #P42, 5 pages. ⇒141
[58] P. Frankl, V. Rödl, Some Ramsey-Turn type results for hypergraphs. *Combin-
batorica* 8, 4 (1988) 323–332. ⇒142
(1992) 309–312. ⇒142
[60] M. Frieze, On the independence number of random graphs, *Discrete Math.* 81,
2 (1990) 171–175. ⇒136
206. ⇒141
[62] Z. Füredi, The number of maximal independent sets in connected graphs, *J.
264–272. ⇒142
240 (2013) 302–324. ⇒142
[65] H. M. Gabow, An efficient implementation of Edmonds’ algorithm for maximum
On vertex independence number of uniform hypergraphs


[77] J. Han, Near perfect matchings in k-uniform hypergraphs, arXiv:1404.1136, 2014, 7 pages. ⇒ 141


113] D. König, Graphs and matrices (Hungarian), Matematikai és Fizikai Lapok 38 (1931) 116–119. $\Rightarrow$ 141


On vertex independence number of uniform hypergraphs


[156] T. Thiele, A lower bound on the independence number of arbitrary hypergraphs, *J. Graph Theory* 30, 3 (1999) 213–221. ⇒ 135

⇒ 141, 142


Received: January 15, 2014 • Revised: May 11, 2014