On multigraphic and potentially multigraphic sequences

Dedicated to the memory of Antal Iványi

Abstract. An $r$-graph (or a multigraph) is a loopless graph in which no two vertices are joined by more than $r$ edges. An $r$-complete graph on $n$ vertices, denoted by $K_n^{(r)}$, is an $r$-graph on $n$ vertices in which each pair of vertices is joined by exactly $r$ edges. A non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ of non-negative integers is said to be $r$-graphic if it is realizable by an $r$-graph on $n$ vertices. An $r$-graphic sequence $\pi$ is said to be potentially $S_{LM}^{(r)}$-graphic if it has a realization containing $S_{LM}^{(r)}$ as a subgraph. We obtain conditions for an $r$-graphic sequence to be potentially $S_{LM}^{(r)}$-graphic. These are generalizations from split graphs to $p$-tuple $r$-split graph.

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1 Introduction

For a positive integer \( r \), an \( r \)-graph(or multigraph) is a loopless graph in which no two vertices are joined by more than \( r \) edges. An \( r \)-complete graph on \( n \) vertices, denoted by \( K_{n}^{(r)} \), is an \( r \)-graph on \( n \) vertices in which each pair of vertices is joined by exactly \( r \) edges. Clearly, \( K_{n}^{(1)} = K_{n} \). A non-increasing sequence \( \pi = (d_{1}, d_{2}, \ldots, d_{n}) \) of non-negative integers is said to be \( r \)-graphic if it is the degree sequence of an \( r \)-graph \( G \) on \( n \) vertices, and such an \( r \)-graph \( G \) is referred to as a realization of \( \pi \). We take \( \sigma(\pi) = \sum_{i=1}^{n} d_{i} \). For graph theoretical notations and definitions we refer to [9].

Let \( \pi = (d_{1}, d_{2}, \ldots, d_{n}) \) be a non-increasing sequence of non-negative integers with \( d_{1} \leq \sum_{i=2}^{n} \min\{r, d_{i}\} \). Define \( \pi'_{k} = (d'_{1}, d'_{2}, \ldots, d'_{n-1}) \) to be the non-increasing rearrangement of the sequence obtained from

\[
(d_{1}, d_{2}, \ldots, d_{k-1}, d_{k+1}, \ldots, d_{n})
\]

by reducing by 1 the remaining largest terms that have not been reduced \( r \) times, and repeating the procedure \( d_{k} \) times. \( \pi'_{k} \) is called the residual sequence obtained from \( \pi \) by laying off \( d_{k} \).

The following three results due to Chungphaisian [2] are generalizations from 1-graphs to \( r \)-graphs of three well-known results, one by Erdős and Gallai [3], one by Kleitman and Wang [6] and one by Fulkerson, Hoffman and Mcandrew [5].

**Theorem 1** [2] Let \( \pi = (d_{1}, d_{2}, \ldots, d_{n}) \) be a non-increasing sequence of non-negative integers, where \( \sigma(\pi) \) is even. Then \( \pi \) is \( r \)-graphic if and only if for each positive integer \( t \leq n \),

\[
\sum_{i=1}^{t} d_{i} \leq rt(t-1) + \sum_{i=t+1}^{n} \min\{rt, d_{i}\}.
\]

**Theorem 2** [2] \( \pi \) is \( r \)-graphic if and only if \( \pi'_{k} \) is \( r \)-graphic.

Let the subgraph \( H \) on the vertices \( v_{1}, v_{j}, v_{k}, v_{l} \) of a multigraph \( G \) contain the edges \( v_{1}v_{j} \) and \( v_{k}v_{l} \). The operation of deleting these edges and introducing a pair of new edges \( v_{1}v_{k} \) and \( v_{j}v_{l} \), or \( v_{j}v_{k} \) and \( v_{1}v_{l} \) is called an elementary degree preserving transformation. If this operation is performed \( r \) times on the same edge set, it is called \( r \)-exchange.
Theorem 3 [2] Let \( \pi \) be an \( r \)-graphic sequence, and let \( G \) and \( G' \) be realizations of \( \pi \). Then there is a sequence of \( r \)-exchanges, \( E_1, \ldots, E_k \) such that the application of these \( r \)-exchanges to \( G \) in order will result in \( G' \).

An \( r \)-graphic sequence \( \pi \) is said to be potentially \( K_{m+1}^{(r)} \) if there exists a realization of \( \pi \) containing \( K_{m+1}^{(r)} \) as a subgraph. If \( \pi \) has a realization \( G \) containing \( K_{m+1}^{(r)} \) on the \( m+1 \) vertices of highest degree in \( G \), then \( \pi \) is said to be potentially \( A_{m+1}^{(r)} \)-graphic. As a special case of Lemma 2.1 in [13], Yin showed that an \( r \)-graphic sequence is potentially \( K_{m+1}^{(r)} \)-graphic if and only if it is potentially \( A_{m+1}^{(r)} \)-graphic.

The \( r \)-join (complete product) of two \( r \)-graphs \( G_1 \) and \( G_2 \) is a graph \( G = G_1 \cup G_2 \) with vertex set \( V(G_1) \cup V(G_2) \) and the edge set consisting of all edges of \( G_1 \) and \( G_2 \) together with the edges joining each vertex of \( G_1 \) with every vertex of \( G_2 \) by exactly \( r \) edges. Let \( K_l^{(r)} \) and \( K_m^{(r)} \) be complete \( r \)-graphs with \( l \) and \( m \) vertices respectively, that is the complete graphs having exactly \( r \) edges between every two vertices. The \( r \)-split graph of \( K_l^{(r)} \) and \( K_m^{(r)} \) denoted by \( S_{l,m}^{(r)} \) is the graph \( K_l^{(r)} \cup K_m^{(r)} \) having \( l + m \) vertices, where \( K_m^{(r)} \) (having no edges) is the complement of \( K_l^{(r)} \). [14]. If \( \pi \) has a realization \( G \) containing \( S_{l,m}^{(r)} \) on the \( l + m \) vertices of highest degree in \( G \), then \( \pi \) is said to be potentially \( A_{l,m}^{(r)} \)-graphic.

The following two results due to Yin [13] are generalizations from 1-graphs to \( r \)-graphs of two well-known results given by A. R. Rao [12].

Theorem 4 [13] Let \( n \geq l + 1 \) and \( \pi = (d_1, d_2, \ldots, d_n) \) be an \( r \)-graphic sequence with \( d_{l+1} \geq rl \). Then \( \pi \) is potentially \( A_{l+1}^{(r)} \)-graphic if and only if \( \pi_{l+1} \) is \( r \)-graphic.

Theorem 5 [13] Let \( n \geq l + 1 \) and \( \pi = (d_1, d_2, \ldots, d_n) \) be an \( r \)-graphic sequence with \( d_{l+1} \geq 2rl - 1 \), then \( \pi \) is potentially \( K_{l+1}^{(r)} \).

An extremal problem for 1-graphic sequences to be potentially \( K_l^{(1)} \)-graphic was considered by Erdős, Jacobson and Lehel [4] and solved by Li et al. [7, 8]. Yin [13] generalized this extremal problem and the Erdős-Jacobson-Lehel conjecture from 1-graphs to \( r \)-graphs.

In 2014, the authors [10] proved the following assertion.
Theorem 6 [10] If \( G_1 \) is a realization of \( \pi_1 = (d_1^1, \ldots, d_m^1) \) containing \( K_p \) as a subgraph and \( G_2 \) is a realization of \( \pi_2 = (d_1^2, \ldots, d_n^2) \) containing \( K_q \) as a subgraph, then the degree sequence \( \pi = (d_1, \ldots, d_{m+n}) \) of the join of \( G_1 \) and \( G_2 \) is potentially \( K_{p+q} \)-graphic.

The following two results for simple graphs are due to Yin [14].

Theorem 7 [14] \( \pi \) is potentially \( \bar{A}_{l,m} \)-graphic if and only if \( \pi_l \) is graphic.

Theorem 8 [14] Let \( n \geq 1 + m \) and let \( \pi = (d_1, d_2, \ldots, d_n) \) be a non-increasing graphic sequence. If \( d_{l+m} \geq 2l + m - 2 \), then \( \pi \) is potentially \( \bar{A}_{l,m} \)-graphic.

A condition for a graphic sequence \( \pi \) to be potentially \( K_4 - e \) graphic can be found in [11], where \( K_4 - e \) is the graph obtained from the complete graph \( K_4 \) by deleting one edge \( e \).

2 Bounds on the sum of squares of degrees of a multigraph

From the Cauchy-Schwarz inequality, we have
\[
\sum_{i=1}^{n} |a_i b_i| \leq \left( \sum_{i=1}^{n} |a_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} |b_i|^2 \right)^{\frac{1}{2}},
\]

Taking \( a_i = d_i \) and \( b_i = 1 \), we have \( \left( \sum_{i=1}^{n} d_i \right)^2 \leq n \sum_{i=1}^{n} d_i^2 \) which implies
\[
\frac{1}{n} \left( \sum_{i=1}^{n} d_i \right)^2 \leq \left( \sum_{i=1}^{n} d_i^2 \right).
\]

From this and the hand shaking Lemma \( \sum_{i=1}^{n} d_i = 2|E| \), we have
\[
\frac{4|E|^2}{n} = \frac{1}{n} \left( \sum_{i=1}^{n} d_i \right)^2 \leq \sum_{i=1}^{n} d_i^2.
\]

Now we have the following observation, the proof is by using the same argument as in Theorem 1 of [1].

Lemma 9 For an \( r \)-graph \( G \), \( \sum_{i=1}^{n} d_i^2 \leq |E|(r(n - 2) + \frac{2|E|}{n-1}) \).
Remark 10 From Lemma 9, we observe that

\[
\frac{4|E|^2}{n} \leq \sum_{i=1}^{n} d_i^2 \leq |E|(n-2) + \frac{2|E|}{n-1}.
\]

The following example shows that the equality does not hold in the above inequality.

Example 11 Consider the 2-graph as shown in Figure 1.

Here, \(\frac{4|E|^2}{n} = \frac{4 \times 16^2}{6} = 3 \frac{12}{5} < 4^2 + 6^2 + 6^2 + 6^2 + 6^2 = 176 < 16(2(6-2) + \frac{2 \times 16}{6-1}) = \frac{152}{3}\).

Now, we have the following result.

Lemma 12 A multigraph \(G\) is regular if and only if \(\frac{4|E|^2}{n} = \sum_{i=1}^{n} d_i^2\).

Proof. Suppose an \(r\)-graph \(G\) is regular of degree \(b\). Then \(2|E| = nb\) and \(d_i = b\) for all \(i = 1, 2, \ldots, n\). We know that \(\sum_{i=1}^{n} d_i^2 = nb^2\) and \(\frac{4|E|^2}{n} = \frac{1}{n}d_1^2 - b^2 = nb^2\).

These together give \(\sum_{i=1}^{n} d_i^2 = \frac{4|E|^2}{n}\).

Conversely, suppose that \(\sum_{i=1}^{n} d_i^2 = \frac{4|E|^2}{n}\). Then \(\frac{4}{n}|E|^2 = \sum_{i=1}^{n} d_i^2\). This implies that

\[
\frac{1}{n}(d_1^2 + d_2^2 + \ldots + d_n^2 + 2(d_1d_2 + d_1d_3 + \ldots + d_1d_n) + \ldots + 2(d_{n-2}d_{n-1} + d_{n-2}d_n) + 2(d_{n-1}d_n)) - (d_1^2 + d_2^2 + \ldots + d_n^2) = 0,
\]

which on simplification gives
A bipartite multigraph

Definition 15

\[ r \text{ Potentially } r \]

\[ 3 \text{ -graphic sequences} \]

Lemma 13 Let \( G \) be an \( r \)-graph with \( n > 2 \) vertices. Then \( G \) is a complete graph \( K_n \) if and only if \( \frac{4|E|^2}{n} = \sum_{i=1}^{n} d_i^2 = |E|(r(n - 2) + \frac{2|E|}{n - 1}) \).

Proof. First we note that an \( r \)-graph \( G \) is a complete \( r \)-graph if and only if \( |E| = \frac{1}{2}rn(n - 1) \). Moreover, we know that \( |E| = \frac{1}{2}nr(n - 1) \), which implies that \( 2|E|(n - 2) + 2|E|n = nr(n - 1)(n - 2) + 2|E|n \) and on simplification gives \( \frac{4|E|^2}{n} = |E|(r(n - 2) + \frac{2|E|}{n - 1}) \). Thus the result follows.

The following result partially answers the question raised in Remark 10.

Theorem 14 A bipartite multigraph \( G = K_n^{(r)} \), where \( m > 1 \), is an \( r \)-star graph \( K_n^{(r)} \) if and only if \( \sum_{i=1}^{n} d_i^2 = |E|(r(n - 2) + \frac{2|E|}{n - 1}) \).

Proof. Let \( K_n^{(r)} \) be an \( r \)-complete bipartite graph, where \( m > 1 \), \( n = l + m \) and \( |E| = rlm \). There are \( l \)-vertices each of whose degree is \( r \times m \) and \( m \) vertices each of whose degree is \( r \times l \), so \( \sum_{i=1}^{n} d_i^2 = l(rm)^2 + m(rl)^2 = lr^2m^2 + mr^2l^2 = r^2(lm^2 + ml^2) = r^2lm(l + m) \). Therefore, we have \( |E|(r(n - 2) + \frac{2|E|}{n - 1}) = rlm(1 + l + m - 2) = r^2lm(l + m) \), which gives \( l = 1 \). Hence the result follows.

3 Potentially \( r \)-graphic sequences

Definition 15 Let \( S_n^{(r)} \), \( S_{r_2,s_2}^{(r)} \), \( S_{r_3,s_3}^{(r)} \), \( S_{r_p,s_p}^{(r)} \) be \( r \)-split graphs, respectively with \( r_1 + s_1 \), \( r_2 + s_2 \), \( r_p + s_p \) vertices. Let \( L = \sum_{i=1}^{p} r_i \) and \( M = \sum_{i=1}^{p} s_i \). Then
the \( p \)-tuple \( r \)-split graph, denoted by \( S_{L,M}^{(r)} \), is the graph

\[
S_{L,M}^{(r)} = \bigoplus_{i=1}^{p} S_{r_{i},s_{i}}^{(r)} = \bigoplus_{i=1}^{p} S_{r_{i},s_{i}}^{(r)} \vee \bigoplus_{i=1}^{p} S_{r_{i},s_{i}}^{(r)}.
\]

Clearly \( S_{L,M}^{(r)} \) has vertex set \( \bigcup_{i=1}^{p} V(S_{r_{i},s_{i}}^{(r)}) \) and the edge set consists of all edges of \( S_{r_{i},s_{i}}^{(r)} \), together with the edges joining each vertex of \( S_{r_{i},s_{i}}^{(r)} \) with every vertex of \( S_{r_{j},s_{j}}^{(r)} \) by exactly \( r \)-edges for every \( i, j \) with \( i \neq j \).

An \( r \)-graphic sequence \( \pi \) is said to be potentially \( S_{L,M}^{(r)} \)-graphic if there exists a realization of \( \pi \) containing \( S_{L,M}^{(r)} \) as a subgraph. If \( \pi \) has a realization \( G \) containing \( S_{L,M}^{(r)} \) on the \( L + M \) vertices of highest degree in \( G \), then \( \pi \) is said to be potentially \( A_{L,M}^{(r)} \)-graphic.

Let \( n \geq L + M \) and let \( \pi = (d_1, \ldots, d_n) \) be a non-increasing sequence of non-negative integers with \( d_L \geq r(L + M) - 1 \) and \( d_{L+M} \geq rL \). We define sequences \( \pi_1, \ldots, \pi_L \) as follows. Construct the sequence

\[
\pi_1 = (d_2 - r, \ldots, d_L - r, d_{L+1} - r, \ldots, d_{L+M} - r, d_{L+M+1}^1, \ldots, d_n^1)
\]

from \( \pi \) by reducing 1 from the largest term that have not been already reduced \( r \) times, and then reordering the last \( n - L - M \) terms to be non-increasing. For \( 2 \leq i \leq r \), construct

\[
\pi_i = (d_{i+1} - ir, \ldots, d_L - ir, d_{L+i-1} - ir, \ldots, d_{L+M} - ir, d_{L+i+1}^i, \ldots, d_n^i)
\]

from

\[
\pi_{i-1} = (d_i - (i-1)r, \ldots, d_L - (i-1)r, d_{L+1} - (i-1)r, \ldots, d_{L+M} - (i-1)r, d_{L+i}^{i-1}, \ldots, d_n^{i-1})
\]

by deleting \( d_i - (i-1)r \), reducing the first \( d_i - (i-1)r \) remaining terms of \( d_{i-1} \) by one that have not been already reduced \( r \) times, and then reordering the last \( n - L - M \) terms to be non-increasing.

We start with the following lemma.
Lemma 16 If $\pi = (d_1, d_2, \ldots, d_m)$ is the graphic sequence of $S^r_{L,M}$, then

$$\pi = \left( \left( \sum_{i=1}^m r(r_i + s_i - 1) \right)^{\tau_i}, \left( \sum_{i=1}^m rr_i + \sum_{i=1, i\neq j}^m rs_i \right)^{s_j} \right), \text{ for } j = 1, 2, \ldots, m.$$ 

Proof. To prove the result we use induction on $m$.

For $m = 1$, the result is obviously true. For $m = 2$, we have $S^r_{1,2}$.

Therefore for every $i = 1, 2, \ldots, r_1$ and $i = 1, 2, 3, \ldots, s_1$ and $j = 1, 2, 3, \ldots, s_2$

$$\bar{d}_i = d_i + r(r_2 + s_2) \eqno(1)$$

and

$$\bar{d}_j = r(r_1 + r_2 + s_2), \eqno(2)$$

where $\bar{d}_i$ and $\bar{d}_j$ are respectively the degree of $\bar{v}_i$ and $\bar{v}_j$ vertex in $S^r_{r_1+r_2,s_1+s_2}$ and $d_i$ is the degree of $i$th vertex in $K_{r_1}$. Equations (1) and (2) hold for every $i, j$. Thus the graphic sequence $\pi^2$ of $S^r_{r_1+r_2, s_1+s_2}$ is

$$\pi^2 = \left( \left( \sum_{i=1}^2 r(r_i + s_i - 1) + r(r_2 + s_2) \right)^{\tau_i}, \left( \sum_{i=1}^2 rr_i + \sum_{i=1, i\neq j}^m rs_i \right)^{s_j} \right).$$

This shows that the result is true for $m = 2$. Assume that the result holds for $m = k - 1$, therefore for all $j = 1, 2, \ldots, k - 1$,

$$\pi^{k-1} = \left( \left( \sum_{i=1}^{k-1} r(r_i + s_i - 1) \right)^{\tau_i}, \left( \sum_{i=1}^{k-1} rr_i + \sum_{i=1, i\neq j}^{k-1} rs_i \right)^{s_j} \right), \text{ for } j = 1, 2.$$ 

Now for $m = k$,

$$G = S^r_{r_1,s_1} \lor S^r_{r_2,s_2} \lor \ldots \lor S^r_{r_{k-1},s_{k-1}} \lor S^r_{r_k,s_k} = A \lor S^r_{r_k,s_k}, \text{ where } A = S^r_{r_1,s_1} \lor S^r_{r_2,s_2} \lor \ldots \lor S^r_{r_{k-1},s_{k-1}}.$$
Since the result is proved for all $m = k - 1$ and using the fact that the result is proved for each pair and since the result is already proved for $k = 2$, it follows by induction hypothesis that result holds for $m = k$ also. That is,

$$
\pi = \left( \left( \sum_{i=1}^{k} r_i (r_i + s_i - 1) \right)^{r_j} \left( \sum_{i=1}^{k} r_i r_j + \sum_{i=1, i \neq j}^{k} r_i s_i \right) \right), \quad \text{for } j = 1, 2, \ldots, k
$$

This proves the lemma. \hfill \Box

**Lemma 17** A non-increasing integer sequence $\pi = (d_1, \ldots, d_n)$ is potentially $A_{L,M}^{(r)}$-graphic if and only if it is potentially $S_{L,M}^{(r)}$-graphic.

**Proof.** We only need to prove that if $\pi = (d_1, \ldots, d_n)$ is potentially $S_{L,M}^{(r)}$-graphic, then it is potentially $A_{L,M}^{(r)}$-graphic. We choose a realization $G$ of $\pi$ with vertex set $V(G) = \{v_1, \ldots, v_n\}$ such that $d_G(v_i) = d_i$ for $1 \leq i \leq n$, the induced $r$-subgraph $G[\{v_1, \ldots, v_{L+M}\}]$ of $\{v_1, \ldots, v_{L+M}\}$ in $G$ contains $S_{L,M}^{(r)}$ as its $r$-subgraph and $|V(K_L^{(r)}) \cap \{v_1, \ldots, v_L\}|$ is maximum. Denote $H = G[\{v_1, \ldots, v_{L+M}\}]$. If $|V(K_L^{(r)}) \cap \{v_1, \ldots, d_i\}| = L$, that is, $V(K_L^{(r)}) = \{v_1, \ldots, v_L\}$, then $\pi$ is potentially $A_{L,M}^{(r)}$-graphic. Assume that $|V(K_L^{(r)}) \cap \{v_1, \ldots, v_L\}| < L$.

Then there exists $v_i \in \{v_1, \ldots, v_L\} \setminus V(K_L^{(r)})$ and a $v_j \in V(K_L^{(r)}) \setminus \{v_1, \ldots, v_L\}$. Let $A = N_H(v_j) \setminus (\{v_i\} \cup N_H(v_i))$ and $B = N_G(v_j) \setminus (\{v_i\} \cup N_G(v_i))$. Since $d_G(v_i) \geq d_G(v_j)$, we have $|B| \geq |A|$. Let $C$ be any subset of $B$ such that $|C| = |A|$. Now form a new realization $G'$ of $\pi$ by a sequence of $r$-exchanges to the $r$-edges of the star centralized at $v_j$ with end vertices in $A$ with the non $r$-edges of the star centralized at $v_j$ with end vertices in $C$, and by a sequence of $r$-exchange the $r$-edges of the star centralized at $v_i$ with end vertices in $C$ with the non $r$-edges of the star centralized at $v_i$ with end vertices in $A$. It is easy to see that $G'$ contains $S_{L,M}^{(r)}$ on $\{v_1, \ldots, v_{L+M}\}$ so that $|V(K_L^{(r)}) \cap \{v_1, \ldots, v_L\}|$ is larger than that of $G$, which contradicts to the choice of $G$. \hfill \Box

We use the Havel-Hakimi procedure to test whether or not an $r$-graphic sequence $\pi$ is potentially $A_{L,M}^{(r)}$-graphic.

**Theorem 18** For $r \geq 1$ and $n \geq 1$, an $r$-graphic sequence $\pi = (d_1, \ldots, d_n)$ is potentially $A_{L,M}^{(r)}$-graphic if and only if $\pi_L$ is $r$-graphic.

**Proof.** Assume that $\pi$ is potentially $A_{L,M}^{(r)}$-graphic. Then $\pi$ has a realization $G$ with the vertex set $V(G) = \{v_1, \ldots, v_n\}$ such that $d_G(v_i) = d_i$ for $(1 \leq i \leq n)$
and $G$ contains $S_{L,M}^{(r)}$ on the vertices $v_1,\ldots,v_{L+M}$, where $L + M \leq n$, so that $V^{(r)}(K_L) = \{v_1,\ldots,v_L\}$ and $V(LM^{(r)}) = \{v_{L+1},\ldots,v_{L+M}\}$. By applying a sequence of $r$-exchanges to $G$ in order we will show that there is one such realization $G'$ such that $G' \setminus v_1$ has degree sequence $\pi_1$. If not, we may choose such a realization $H$ of $r$-graphic sequence $\pi$ such that the number of vertices adjacent to $v_1$ in $\{v_{L+M+1},\ldots,v_{d+1}\}$ is maximum. Let $v_i \in \{v_{L+M+1},\ldots,v_{d+1}\}$ and assume that there is no edge between $v_1$ and $v_i$ and let $v_j \in \{v_{d+2},\ldots,v_n\}$ and there are $r$ edges between $v_1$ and $v_j$. We may assume that $d_i > d_j$. Hence there is a vertex $v_t, t \neq i, j$ such that there are $r$ edges between $v_t$ and $v_1$ and no edge between $v_t$ and $v_j$. Clearly $G = (H \setminus (v_i^{(r)}v_j, v_i^{(r)}v_t)) \cup (v_i^{(r)}v_t, v_i^{(r)}v_j)$ (where $v_i^{(r)}v_j$ means that there are $r$ edges between $v_i$ and $v_j$) is a realization of $\pi$ such that $d_G(v_i) = d_i$ for $1 \leq i \leq n$, $G$ contains $S_{L,M}^{(r)}$ on $v_1,\ldots,v_{L+M}$ with $V^{(r)}(K_L) = \{v_1,\ldots,v_L\}$ and $V(LM^{(r)}) = \{v_{L+1},\ldots,v_{L+M}\}$ and $H$ has the number of vertices adjacent to $v_1$ in $\{v_{L+M+1},\ldots,v_{d+1}\}$ larger than that of $H$. This contradicts the choice of $H$. Repeating this procedure, we can see that $\pi_t$ is potentially $A_{L-M}^{(r)}$-graphic successively for $i = 2,\ldots,L$. In particular, $\pi_L$ is $r$-graphic.

Conversely, suppose that $\pi_t$ is $r$-graphic and is realized by a graph $G_t$ with a vertex set $V(G_t) = \{v_{L+1},\ldots,v_n\}$ such that $d_{G_t}(v_i) = d_i$ for $1 \leq i \leq n$. For $i = L, L-1,\ldots,1$ form $G_{t-1}$ from $G_i$ by adding a new vertex $v_i$ that is adjacent to each of $v_{i+1},\ldots,v_{L+M}$ with $r$-edges and also to the vertices of $G_i$ with degrees $\frac{s_{i-1}^{L+M+i-1}-r}{r},\ldots,\frac{d_{i+1}^{L+M+i-1}-r}{r}$. Then for each $i$, $G_i$ has degrees given by $\pi_i$ and $G_i$ contains $S_{L-M}^{(r)}$ on $L+M-i$ vertices $v_{i+1},\ldots,v_{L+M}$ whose degrees are $d_i + r,\ldots,d_L + r$ so that $V(K_{L-M}^{(r)}) = \{v_{i+1},\ldots,v_L\}$ and $V(K_{L+M}^{(r)}) = \{v_{L+1},\ldots,v_{L+M}\}$. In particular, $G_0$ has degrees given by $\pi$ and contains $S_{L,M}^{(r)}$ on $L+M$ vertices $v_1,\ldots,v_{L+M}$ whose degrees are $d_1,\ldots,d_L + r$ so that $V(K_L^{(r)}) = \{v_1,\ldots,v_L\}$ and $V(K_{L+M}^{(r)}) = \{v_{L+1},\ldots,v_{L+M}\}$. Hence the result follows.

The following is a sufficient condition for an $r$-graphic sequence to be potentially $A_{L,M}^{(r)}$-graphic.

**Theorem 19** Let $n \geq L + M$ and let $\pi = (d_1,\ldots,d_n)$ be an $r$-graphic sequence. If $d_{L+M} \geq 2rL + rM - 2$, then $\pi$ is potentially $A_{L,M}^{(r)}$-graphic.

**Proof.** Let $n \geq L + M$ and let $\pi = (d_1,\ldots,d_n)$ be a non-increasing $r$-graphic sequence with $d_{L+M} \geq 2rL + rM - 2$. By using the argument similar
to Theorem 8, \( \pi \) is potentially \( K_{L}^{(r)} \)-graphic and hence by Lemma 17, \( A_{L}^{(r)} \)-graphi
c. Therefore, we assume that \( G \) is a realization of \( \pi \) with a vertex set \( V(G) = \{v_1, \ldots, v_n\} \) such that \( d_{G}(v_i) = d_i, \ (1 \leq i \leq n) \) and \( G \) contains \( K_{L}^{(r)} \) on \( \{v_1, \ldots, v_L\} \), that is, \( V(K_{L}^{(r)}) = \{v_1, \ldots, v_L\} \) and

\[
t = e_{G}(\{v_1, \ldots, v_{rL}, \ldots, v_{L+1}, \ldots, v_{L+s_1}, \ldots, v_{L+M}\})
\]

(that is, the number of edges between \( \{v_1, \ldots, v_L\} \) and \( \{v_{L+1}, \ldots, v_{L+M}\} \)) is maximum. If \( t = rLM + rs_{1}s_{2} + s_{j} \sum_{i=1}^{j-1} rs_{i} \), for \( j = 3, 4, \ldots, p \), then the cardinality of the edge set of \( S_{L,M}^{(r)} \) is same as \( t \) and therefore \( G \) contains \( S_{L,M}^{(r)} \) on the vertices \( v_{1}, v_{2}, \ldots, v_{L+M} \) with \( V[\{K_{M}\}] = \{v_{1}, v_{2}, \ldots, v_{L}\} \) and

\[
V(\bar{K}_{M}^{(r)}) = \{v_{L+1}, v_{L+2}, \ldots, v_{L+s_{1}}, \ldots, v_{L+M}\}.
\]

In other-words, \( \pi \) is potentially \( \bar{A}_{L,M}^{(r)} \)-graphic. Assume that \( t < rLM + rs_{1}s_{2} + s_{j} \sum_{i=1}^{j-1} rs_{i} \), for \( j = 3, 4, \ldots, p \). Then there exists a \( v_{k} \in \{v_{1}, v_{2}, \ldots, v_{s_{1}}\} \) and \( v_{m} \in \{v_{s_{1}+1}, v_{s_{1}+2}, \ldots, v_{s_{1}+s_{2}}\}, (i \neq j) \) such that \( v_{k}v_{m} \notin E(G) \). Let

\[
A = N_{G}\backslash v_{s_{1}+1}, v_{s_{1}+2}, \ldots, v_{s_{1}+s_{2}}(v_{k}) \setminus N_{G}\backslash v_{1}, v_{2}, \ldots, v_{s_{1}}(v_{m})
\]

and

\[
B = N_{G}\backslash v_{s_{1}+1}, v_{s_{1}+2}, \ldots, v_{s_{1}+s_{2}}(v_{k}) \cap N_{G}\backslash v_{1}, v_{2}, \ldots, v_{s_{1}}(v_{m}).
\]

Then \( e_{G}(x, y) = r \) for \( x \in N_{G}\backslash v_{1}, \ldots, v_{L}(v_{m}) \) and \( y \in N_{G}\backslash v_{L+1}, \ldots, v_{L+M}(v_{k}) \). Otherwise, if \( e_{G}(x, y) < r \), then \( G' = (G \setminus \{v_{m}^{(r)}y, v_{m}^{(r)}x\}) \cup \{v_{k}^{(r)}v_{m}, x^{(r)}y\} \) is a realization of \( \pi \) and contains \( \bar{S}_{L,M}^{(r)} \) on \( v_{1}, \ldots, v_{L+M} \) with \( V(K_{M}^{(r)}) = \{v_{1}, \ldots, v_{L}\} \) and \( (\bar{K}_{M}^{(r)}) = \{v_{L+1}, \ldots, v_{L+M}\} \) such that

\[
e_{G'}(\{v_{1}, \ldots, v_{L}\}, \{v_{L+1}, \ldots, v_{L+M}\}) > t,
\]

which contradicts the choice of \( G \). Thus \( B \) is \( r \)-complete. We consider the following cases.

Let \( A = \emptyset \). Then \( 2rL + rM - 2 \leq d_{k} = d_{G}(v_{k}) < rL + rM - 1 + r|B| \), and so \( |B| \geq rL \). Since each vertex in \( N_{G}\backslash v_{1}, \ldots, v_{L}(v_{m}) \) is adjacent to each vertex in \( B \) by \( r \) edges and \( |N_{G}\backslash v_{1}, \ldots, v_{L}(v_{m})| \geq 2rL + rM - 2 = rL + rM - 1 \). It can be easily seen that the \( r \) induced subgraph of \( N_{G}\backslash v_{1}, \ldots, v_{L}(v_{m}) \cup \{v_{m}\} \) in \( G \) contains \( \bar{S}_{L,M}^{(r)} \).
as a subgraph. Thus \( \pi \) is potentially \( \overline{A}_{LM}^{(r)} \)-graphic.

Let \( A \neq \emptyset \). Let \( a \in A \). If there are \( x, y \in N_{G \setminus \{v_1, \ldots, v_L\}}(v_m) \) such that \( e_G(x, y) < r \) then \( G' = (G \setminus \{v_m, a\}) \cup \{v_k, v_m, x, y\} \) is a realization of \( \pi \) and contains \( \overline{S}_{LM}^{(r)} \) on \( v_1, \ldots, v_L \) with \( V(K_L^{(r)}) = \{v_1, \ldots, v_L\} \) and \( V(K_M^{(r)}) = \{v_{L+1}, \ldots, v_{L+M}\} \) such that \( e_{G'}(\{v_1, \ldots, v_L\}, \{v_{L+1}, \ldots, v_{L+M}\}) > t \) which contradicts the choice of \( G \). Thus \( N_{G \setminus \{v_1, \ldots, v_L\}}(v_m) \) is \( r \)-complete. Since

\[ |N_{G \setminus \{v_1, \ldots, v_L\}}(v_m)| \geq rL + rM - 1 \quad \text{and} \quad e_{G}(v_m, z) = r, \]

for any \( z \in N_{G \setminus \{v_1, \ldots, v_L\}}(v_m) \), it is easy to see that the induced \( r \)-subgraph of \( N_{G \setminus \{v_1, \ldots, v_L\}}(v_m) \cup \{v_m\} \) in \( G \) is \( r \)-complete, and so contains \( \overline{S}_{LM}^{(r)} \) as a \( r \)-subgraph. Thus \( \pi \) is potentially \( \overline{A}_{LM}^{(r)} \)-graphic. \( \square \)

**Theorem 20** If \( \pi = (d_1, d_2, \ldots, d_n) \) is an \( r \)-graphic sequence such that \( \sigma(\pi) \) is at least \( (n^2 - 3n + 8) r \), then \( \pi \) is potentially \( K_4^{(r)} \)-graphic.

**Proof.** Let \( \pi = (d_1, d_2, \ldots, d_n) \) be an \( r \)-graphic sequence such that \( d_1 \geq d_2 \geq \ldots \geq d_n \geq 1 \) and \( \sigma(\pi) = (n^2 - 3n + 8) r \). Suppose \( G \) is a graphical realization of \( \pi \) and \( e(G) \) is the size of \( G \). Then \( 2e(G) = \sigma(\pi) \) and \( 2e(G^c) = nb(n - 1) - \sigma(\pi) = nr(n - 1) - (n^2 - 3n + 6)r = r(2n - 6) \), so that \( e(G^c) = r(n - 3) \), where \( G^c \) is the complement of the \( r \)-graph \( G \). An extremal problem is \( r \)-graph \( G \) is obtained by deleting \( r(n - 3) \) independent edges from the complete \( r \)-graph \( K_n^{(r)} \) of order \( n \). Hence the largest vertex number of independent sets in \( G^c \) is 3. This implies that the largest possible complete \( r \)-subgraph of \( G \) is of order 3. As \( 1 \leq n - 3 \leq 3 \). Hence there is no complete \( r \)-subgraph of order 4 in \( G \). Therefore, we have

\[ \sigma(K_4^{(r)}, n) \geq (n^2 - 3n + 6)r + 2r = (n^2 - 3n + 8)r \]

Now Suppose that \( \pi = (d_1, d_2, \ldots, d_n) \) is \( r \)-graphic sequence with \( d_1 \geq d_2 \geq \ldots \geq d_n \geq r \) and \( \sigma(\pi) \geq (n^2 - 3n + 8) r \). Then every graphical realization \( G \) of \( \pi \) is obtained by removing at most \( r(n - 4) \) edges from the \( r \)-complete graph \( K_n^{(r)} \). Hence the maximal complete subgraph of \( G \) has order at least \( n - (n - 4) = 4 \). Thus \( G \) is potentially \( K_4^{(r)} \). In other words,

\[ \sigma(K_4^{(r)}, n) \leq (n^2 - 3n + 8)r \quad (3) \]

Combining (3) and (4), the result follows. \( \square \)

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