Imbalances in directed multigraphs

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Abstract. In a directed multigraph, the imbalance of a vertex $v_i$ is defined as $b_{v_i} = d_{v_i}^+ - d_{v_i}^-$, where $d_{v_i}^+$ and $d_{v_i}^-$ denote the outdegree and indegree respectively of $v_i$. We characterize imbalances in directed multigraphs and obtain lower and upper bounds on imbalances in such digraphs. Also, we show the existence of a directed multigraph with a given imbalance set.

1 Introduction

A directed graph (shortly digraph) without loops and without multi-arcs is called a simple digraph [2]. The imbalance of a vertex $v_i$ in a digraph as $b_{v_i}$ (or simply $b_i$) = $d_{v_i}^+ - d_{v_i}^-$, where $d_{v_i}^+$ and $d_{v_i}^-$ are respectively the outdegree and indegree of $v_i$. The imbalance sequence of a simple digraph is formed by listing the vertex imbalances in non-increasing order. A sequence of integers $F = [f_1, f_2, \ldots, f_n]$ with $f_1 \geq f_2 \geq \ldots \geq f_n$ is feasible if the sum of its elements is zero, and satisfies $\sum_{i=1}^{k} f_i \leq k(n - k)$, for $1 \leq k < n$.

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The following result [5] provides a necessary and sufficient condition for a sequence of integers to be the imbalance sequence of a simple digraph.

**Theorem 1** A sequence is realizable as an imbalance sequence if and only if it is feasible.

The above result is equivalent to saying that a sequence of integers $B = [b_1, b_2, \ldots, b_n]$ with $b_1 \geq b_2 \geq \ldots \geq b_n$ is an imbalance sequence of a simple digraph if and only if

$$\sum_{i=1}^{k} b_i \leq k(n - k),$$

for $1 \leq k < n$, with equality when $k = n$.

On arranging the imbalance sequence in non-decreasing order, we have the following observation.

**Corollary 1** A sequence of integers $B = [b_1, b_2, \ldots, b_n]$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ is an imbalance sequence of a simple digraph if and only if

$$\sum_{i=1}^{k} b_i \geq k(k - n),$$

for $1 \leq k < n$ with equality when $k = n$.

Various results for imbalances in simple digraphs and oriented graphs can be found in [6], [7].

## 2 Imbalances in $r$-graphs

A multigraph is a graph from which multi-edges are not removed, and which has no loops [2]. If $r \geq 1$ then an $r$-digraph (shortly $r$-graph) is an orientation of a multigraph that is without loops and contains at most $r$ edges between the elements of any pair of distinct vertices. Clearly $1$-digraph is an oriented graph.

Let $D$ be an $f$-digraph with vertex set $\{v_1, v_2, \ldots, v_n\}$, and let $d^+_v$ and $d^-_v$ respectively denote the outdegree and indegree of vertex $v$. Define $b_{v_i}$ (or simply $b_i$) as imbalance of $v_i$. Clearly, $-r(n-1) \leq b_{v_i} \leq r(n-1)$. The imbalance sequence of $D$ is formed by listing the vertex imbalances in non-decreasing order.
We remark that r-digraphs are special cases of (a, b)-digraphs containing at least a and at most b edges between the elements of any pair of vertices. Degree sequences of (a, b)-digraphs are studied in [3, 4].

Let u and v be distinct vertices in D. If there are f arcs directed from u to v and g arcs directed from v to u, we denote this by u(f − g)v, where 0 ≤ f, g, f + g ≤ r.

A double in D is an induced directed subgraph with two vertices u, v having the form u(f_1f_2)v, where 1 ≤ f_1, f_2 ≤ r, and 1 ≤ f_1 + f_2 ≤ r, and f_1 is the number of arcs directed from u to v, and f_2 is the number of arcs directed from v to u. A triple in D is an induced subgraph with tree vertices u, v, and w having the form u(f_1f_2)v(g_1g_2)w(h_1h_2)u, where 1 ≤ f_1, f_2, g_1, g_2, h_1, h_2 ≤ r, and 1 ≤ f_1 + f_2, g_1 + g_2, h_1 + h_2 ≤ r, and the meaning of f_1, f_2, g_1, g_2, h_1, h_2 is similar to the meaning in the definition of doubles. An oriented triple in D is an induced subdigraph with three vertices. An oriented triple is said to be transitive if it is of the form u(1 − 0)v(1 − 0)w(0 − 1)u, or u(1 − 0)v(0 − 0)u, or u(1 − 0)v(0 − 0)w(0 − 1)u, or u(1 − 0)v(0 − 0)w(0 − 0)u, otherwise it is intransitive. An r-graph is said to be transitive if all its oriented triples are transitive. In particular, a triple C in an r-graph is transitive if every oriented triple of C is transitive.

The following observation can be easily established and is analogues to Theorem 2.2 of Avery [1].

**Lemma 1** If D_1 and D_2 are two r-graphs with same imbalance sequence, then D_1 can be transformed to D_2 by successively transforming (i) appropriate oriented triples in one of the following ways, either (a) by changing the intransitive oriented triple u(1 − 0)v(1 − 0)w(1 − 0)u to a transitive oriented triple u(0 − 0)v(0 − 0)w(0 − 0)u, which has the same imbalance sequence or vice versa, or (b) by changing the intransitive oriented triple u(1 − 0)v(1 − 0)w(0 − 0)u to a transitive oriented triple u(0 − 0)v(0 − 0)w(0 − 1)u, which has the same imbalance sequence or vice versa; or (ii) by changing a double u(1 − 1)v to a double u(0 − 0)v, which has the same imbalance sequence or vice versa.

The above observations lead to the following result.

**Theorem 2** Among all r-graphs with given imbalance sequence, those with the fewest arcs are transitive.

**Proof.** Let B be an imbalance sequence and let D be a realization of B that is not transitive. Then D contains an intransitive oriented triple. If it is of
the form $u(1-0)v(1-0)w(1-0)u$, it can be transformed by operation $i(a)$ of Lemma 3 to a transitive oriented triple $u(0-0)v(0-0)w(0-0)u$ with the same imbalance sequence and three arcs fewer. If $D$ contains an intransitive oriented triple of the form $u(1-0)v(1-0)w(0-0)u$, it can be transformed by operation $i(b)$ of Lemma 3 to a transitive oriented triple $u(0-0)v(0-0)w(0-1)u$ with the same imbalance sequence but one arc fewer. In case $D$ contains both types of intransitive oriented triples, they can be transformed to transitive ones with certainly lesser arcs. If in $D$ there is a double $u(1-1)v$, by operation $(ii)$ of Lemme 4, it can be transformed to $u(0-0)v$, with same imbalance sequence but two arcs fewer. □

The next result gives necessary and sufficient conditions for a sequence of integers to be the imbalance sequence of some $r$-graph.

**Theorem 3** A sequence $B = [b_1, b_2, \ldots, b_n]$ of integers in non-decreasing order is an imbalance sequence of an $r$-graph if and only if

$$\sum_{i=1}^{k} b_i \geq rk(k-n),$$

with equality when $k = n$.

**Proof. Necessity.** A multi subdigraph induced by $k$ vertices has a sum of imbalances $rk(k-n)$.

**Sufficiency.** Assume that $B = [b_1, b_2, \ldots, b_n]$ be the sequence of integers in non-decreasing order satisfying conditions (1) but is not the imbalance sequence of any $r$-graph. Let this sequence be chosen in such a way that $n$ is the smallest possible and $b_1$ is the least with that choice of $n$. We consider the following two cases.

**Case (i).** Suppose equality in (1) holds for some $k \leq n$, so that

$$\sum_{i=1}^{k} b_i = rk(k-n),$$

for $1 \leq k < n$.

By minimality of $n$, $B_1 = [b_1, b_2, \ldots, b_k]$ is the imbalance sequence of some $r$-graph $D_1$ with vertex set, say $V_1$. Let $B_2 = [b_{k+1}, b_{k+2}, \ldots, b_n]$. Consider,
\[
\sum_{i=1}^{f} b_{k+i} = \sum_{i=1}^{k+f} b_i - \sum_{i=1}^{k} b_i \\
\geq r(k+f)([k+f] - n) - rk(k-n) \\
= r(k^2 + kf - kn + fk + f^2 - fn - k^2 + kn) \\
\geq r(f^2 - fn)
\]

for \(1 \leq f \leq n-k\), with equality when \(f = n-k\). Therefore, by the minimality for \(n\), the sequence \(B_2\) forms the imbalance sequence of some \(r\)-graph \(D_2\) with vertex set, say \(V_2\). Construct a new \(r\)-graph \(D\) with vertex set as follows.

Let \(V = V_1 \cup V_2\) with, \(V_1 \cap V_2 = \emptyset\) and the arc set containing those arcs which are in \(D_1\) and \(D_2\). Then we obtain the \(r\)-graph \(D\) with the imbalance sequence \(B\), which is a contradiction.

**Case (ii).** Suppose that the strict inequality holds in (1) for some \(k < n\), so that

\[
\sum_{i=1}^{k} b_i > rk(k-n),
\]

for \(1 \leq k < n\). Let \(B_1 = [b_1 - 1, b_2, \ldots, b_{n-1}, b_n + 1]\), so that \(B_1\) satisfy the conditions (1). Thus by the minimality of \(b_1\), the sequences \(B_1\) is the imbalances sequence of some \(r\)-graph \(D_1\) with vertex set, say \(V_1\). Let \(b_{v_1} = b_1 - 1\) and \(b_{v_n} = a_n + 1\). Since \(b_{v_n} > b_{v_1} + 1\), there exists a vertex \(v_p \in V_1\) such that \(v_n(0-0)v_p(1-0)v_1\), or \(v_n(1-0)v_p(0-0)v_1\), or \(v_n(1-0)v_p(1-0)v_1\), or \(v_n(0-0)v_p(0-0)v_1\), and if these are changed to \(v_n(0-1)v_p(0-0)v_1\), or \(v_n(0-0)v_p(0-1)v_1\), or \(v_n(0-0)v_p(0-0)v_1\), or \(v_n(0-1)v_p(0-1)v_1\) respectively, the result is an \(r\)-graph with imbalances sequence \(B\), which is again a contradiction. This proves the result.

Arranging the imbalance sequence in non-increasing order, we have the following observation.

**Corollary 2** A sequence \(B = [b_1, b_2, \ldots, b_n]\) of integers with \(b_1 \geq b_2 \geq \ldots \geq b_n\) is an imbalance sequence of an \(r\)-graph if and only if

\[
\sum_{i=1}^{k} b_i \leq rk(n-k),
\]

for \(1 \leq k \leq n\), with equality when \(k = n\).
The converse of an $r$-graph $D$ is an $r$-graph $D'$, obtained by reversing orientations of all arcs of $D$. If $B = [b_1, b_2, \ldots, b_n]$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ is the imbalance sequence of an $r$-graph $D$, then $B' = [-b_n, -b_{n-1}, \ldots, -b_1]$ is the imbalance sequence of $D$.

The next result gives lower and upper bounds for the imbalance $b_i$ of a vertex $v_i$ in an $r$-graph $D$.

**Theorem 4** If $B = [b_1, b_2, \ldots, b_n]$ is an imbalance sequence of an $r$-graph $D$, then for each $i$

$$r(i - n) \leq b_i \leq r(i - 1).$$

**Proof.** Assume to the contrary that $b_i < r(i - n)$, so that for $k < i$,

$$b_k \leq b_i < r(i - n).$$

That is,

$$b_1 < r(i - n), b_2 < r(i - n), \ldots, b_i < r(i - n).$$

Adding these inequalities, we get

$$\sum_{k=1}^{i} b_k < ri(i - n),$$

which contradicts Theorem 3.

Therefore, $r(i - n) \leq b_i$.

The second inequality is dual to the first. In the converse $r$-graph with imbalance sequence $B = [b_1', b_2', \ldots, b_n']$ we have, by the first inequality

$$b_{n-i+1}' \geq r[(n - i + 1) - n]$$

$$= r(-i + 1).$$

Since $b_1 = -b_{n-i+1}'$, therefore

$$b_i \leq -r(-i + 1) = r(i - 1).$$

Hence, $b_i \leq r(i - 1)$. \qed

Now we obtain the following inequalities for imbalances in $r$-graphs.

**Theorem 5** If $B = [b_1, b_2, \ldots, b_n]$ is an imbalance sequence of an $r$-graph with $b_1 \geq b_2 \geq \ldots \geq b_n$, then

$$\sum_{i=1}^{k} b_i^2 \leq \sum_{i=1}^{k} (2rn - 2rk - b_i)^2,$$

for $1 \leq k \leq n$ with equality when $k = n$. 
Proof. By Theorem 3, we have for \(1 \leq k \leq n\) with equality when \(k = n\)

\[ rk(n - k) \geq \sum_{i=1}^{k} b_i, \]

implying

\[ \sum_{i=1}^{k} b_i^2 + 2(2rn - 2rk)rk(n - k) \geq \sum_{i=1}^{k} b_i^2 + 2(2rn - 2rk) \sum_{i=1}^{k} b_i, \]

from where

\[ \sum_{i=1}^{k} b_i^2 + k(2rn - 2rk)^2 - 2(2rn - 2rk) \sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k} b_i^2, \]

and so we get the required

\[ b_1^2 + b_2^2 + \ldots + b_k^2 + (2rn - 2rk)^2 + (2rn - 2rk)^2 + \ldots + (2rn - 2rk)^2 - 2(2rn - 2rk)b_1 - 2(2rn - 2rk)b_2 - \ldots - 2(2rn - 2rk)b_k \]

\[ \geq \sum_{i=1}^{k} b_i^2, \]

or

\[ \sum_{i=1}^{k} (2rn - 2rk - b_i)^2 \geq \sum_{i=1}^{k} b_i^2. \]

The set of distinct imbalances of vertices in an \(r\)-graph is called its imbalance set. The following result gives the existence of an \(r\)-graph with a given imbalance set. Let \((p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n)\) denote the greatest common divisor of \(p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n\).

Theorem 6 If \(P = \{p_1, p_2, \ldots, p_m\}\) and \(Q = \{-q_1, -q_2, \ldots, -q_n\}\) where \(p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n\) are positive integers such that \(p_1 < p_2 < \ldots < p_m\) and \(q_1 < q_2 < \ldots < q_n\) and \((p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n) = t, 1 \leq t \leq r\), then there exists an \(r\)-graph with imbalance set \(P \cup Q\).

Proof. Since \((p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n) = t, 1 \leq t \leq r\), there exist positive integers \(f_1, f_2, \ldots, f_m\) and \(g_1, g_2, \ldots, g_n\) with \(f_1 < f_2 < \ldots < f_m\) and \(g_1 < g_2 < \ldots < g_n\) such that

\[ p_i = tf_i. \]
for \(1 \leq i \leq m\) and
\[ q_i = tg_i \]
for \(1 \leq j \leq n\).

We construct an \(r\)-graph \(D\) with vertex set \(V\) as follows.

Let
\[ V = X_1^1 \cup X_2^1 \cup \ldots \cup X_m^1 \cup X_1^2 \cup X_2^2 \cup \ldots \cup X_m^2 \cup \ldots \cup X_1^n \cup X_2^n \cup \ldots \cup X_m^n \cup Y_1^1 \cup Y_2^1 \cup \ldots \cup Y_m^1 \cup Y_1^2 \cup Y_2^2 \cup \ldots \cup Y_m^2 \cup \ldots \cup Y_1^n, \]
with \(X_i^j \cap X_k^l = \emptyset\), \(Y_i^j \cap Y_k^l = \emptyset\), \(X_i^j \cap Y_k^l = \emptyset\) and
\[ |X_i^j| = g_i, \text{ for all } 1 \leq i \leq m, \]
\[ |X_i^j| = g_i, \text{ for all } 2 \leq i \leq n, \]
\[ |Y_i^j| = f_i, \text{ for all } 1 \leq i \leq m, \]
\[ |Y_i^j| = f_i, \text{ for all } 2 \leq i \leq n. \]

Let there be \(t\) arcs directed from every vertex of \(X_i^1\) to each vertex of \(Y_i^1\), for all \(1 \leq i \leq m\) and let there be \(t\) arcs directed from every vertex of \(X_i^j\) to each vertex of \(Y_i^j\), for all \(2 \leq i \leq n\) so that we obtain the \(r\)-graph \(D\) with imbalances of vertices as under.

For \(1 \leq i \leq m\), for all \(x_1^1 \in X_1^1\)
\[ b_{x_1^1} = t|Y_1^1| - 0 = tf_1 = p_1, \]
for \(2 \leq i \leq n\), for all \(x_1^i \in X_1^i\)
\[ b_{x_1^i} = t|Y_1^i| - 0 = tf_1 = p_1, \]
for \(1 \leq i \leq m\), for all \(y_1^1 \in Y_1^1\)
\[ b_{y_1^1} = 0 - t|X_1^1| = -tg_1 = -q_1, \]
and for \(2 \leq i \leq n\), for all \(y_1^i \in Y_1^i\)
\[ b_{y_1^i} = 0 - t|X_1^i| = -tg_1 = -q_1. \]

Therefore imbalance set of \(D\) is \(P \cup Q\). □

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