On a structured semidefinite program

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Abstract. The nonnegative biquadratics discussed comes from the Böttcher-Wenzel inequality. It is for some matrices a sum of squares of polynomials (SOS), in other cases not, depending on the nonzero pattern of the matrices at issue. Our aim was to draw a line between them. To prove the ‘not a SOS’ case we solve a semidefinite programming (SDP) problem. Subsequently a two-parameter version will be investigated.

1 Introduction

The Böttcher-Wenzel inequality (see [2], [7], [3], [1], [9], [5], [4], [10]) states (in its stronger form) that for real square matrices $X, Y$ of the same order $n$

$$f(X, Y) = 2 \|X\|^2 \|Y\|^2 - 2 \text{trace}^2(X^TY) - \|XY - YX\|^2 \geq 0,$$  

where the norm used is the Frobenius norm. Since all our attempts to obtain a representation for $f$ as a sum of polynomial squares (in short: SOS) failed for $n = 3$, distinguishing between the ‘good’ and ‘bad’ cases became to a natural problem.

In case of $n = 2$ we have for $X = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 & y_3 \\ y_2 & y_4 \end{pmatrix}$ and with variables $z_{i,j} = x_i y_j - y_i x_j$, $1 \leq i < j \leq 4$, that

$$f(X, Y) = 2 z_{1,4}^2 + (z_{1,2} - z_{2,4})^2 + (z_{1,3} - z_{3,4})^2$$

is a sum of squares of quadratics.

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Our main result is that the nonnegative form (1) is SOS for good matrices $X, Y$, whereas it isn’t SOS for general bad matrices, where a matrix of order $n$ will be called good, if nonzero elements occur only in row 1 and column $n$, while it is called bad, if, moreover, nonzero elements occur also in the main diagonal, as shown e.g. for $n = 4$:

\[
\text{good} : \begin{pmatrix} * & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix} \quad \text{bad} : \begin{pmatrix} * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.
\]

**Remark 1** For convenience we re-cite the SOS representation [8] for the good cases in section 2. Section 3 contains the main result: the non-possibility of an SOS-representation for the bad cases via SDP, while in section 4 we provide the function $f$ with two parameters and decompose the unit square into regions with different properties.

## 2 SOS decomposition for good matrices

Let $X, Y$ be good real $n$-th order matrices with $m = 2n - 1$ possible nonzero elements:

\[
X = \begin{pmatrix} x_1 & \ldots & x_{n-1} & x_n \\ 0 & \ldots & 0 & x_{n+1} \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & x_m \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & \ldots & y_{n-1} & y_n \\ 0 & \ldots & 0 & y_{n+1} \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & y_m \end{pmatrix},
\]

and define an $m$-th order matrix $Z$ by help of vectors $x = (x_i)^{m}_{i=1}$ and $y = (y_i)^{m}_{i=1}$ as

\[
Z = xy^T - yx^T = (z_{i,j})^{m}_{i,j=1}, \quad z_{i,j} = x_i y_j - y_i x_j.
\]

The SOS representation for these good matrices is the following.
Theorem 1 (Theorem 1, [8])

\[
\|Z\|^2 - \left( \sum_{i=1}^{n} z_{i,i+n-1} \right)^2 - \sum_{i=2}^{n-1} z_{1,i}^2 - \sum_{i=n+1}^{m-1} z_{i,m}^2
\]

\[= \sum_{i=1}^{n-1} \sum_{j=n+1}^{m} z_{i,j}^2 + \sum_{i=2}^{n-1} \sum_{j=i+1}^{n-1} z_{i,j}^2 + \sum_{i=n+1}^{2n-3} \sum_{j=i+1}^{2n-2} z_{i,j}^2
\]

\[+ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (z_{i,j} - z_{i+n-1,j+n-1})^2 + \sum_{i=1}^{n} \sum_{j=i+1}^{n} (z_{i,j+n-1} - z_{j,i+n-1})^2.
\]

Remark 2 Indeed, (1) and (2) are identical. In particular,

\[\|Z\|^2 = 2 \|X\|^2 \|Y\|^2 - 2 \text{trace}^2(X^TY),\]

and

\[\left( \sum_{i=1}^{n} z_{i,i+n-1} \right)^2 + \sum_{i=2}^{n-1} z_{1,i}^2 + \sum_{i=n+1}^{2n-1} z_{i,m}^2 = \|XY - YX\|^2\]

holds, where the first is Lagrange's identity, the second is straightforward.

3 SOS decomposition impossible for bad matrices

It suffices to prove this negative result for third order matrices. Let

\[X = \begin{pmatrix} x_1 & x_2 & x_3 \\ 0 & x_6 & x_4 \\ 0 & 0 & x_5 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & y_2 & y_3 \\ 0 & y_6 & y_4 \\ 0 & 0 & y_5 \end{pmatrix}.
\]

It turns out that the presence of 6 and y_6 causes the impossibility of an SOS representation for (1). Since

\[XY - YX = \begin{pmatrix} 0 & z_{1,2} + z_{2,6} & z_{1,3} + z_{2,4} + z_{3,5} \\ 0 & 0 & z_{4,5} - z_{4,6} \\ 0 & 0 & 0 \end{pmatrix},\]

the nonnegative form (1) to be discussed assumes the form

\[2 \sum_{1 \leq i < j \leq 6} z_{i,j}^2 - (z_{1,3} + z_{2,4} + z_{3,5})^2 - (z_{1,2} + z_{2,6})^2 - (z_{4,5} - z_{4,6})^2 \]

with \(z_{i,j} = x_i y_j - x_j y_i, 1 \leq i < j \leq 6.\)
Theorem 2 The biquadratic form (3), nonnegative for any real \((x_i)^6, (y_i)^6\),
is not a sum of squares of any quadratics!

We will need a lemma before proving the theorem.

Lemma 1 An sos-representation of \(f\) is necessarily a sum of squares of the \(z_{i,j}\)’s. Furthermore, for the variables \(z_{i,j}\) the basic identities
\[
z_{i,j} z_{k,l} + z_{i,l} z_{j,k} - z_{i,k} z_{j,l} = 0, \quad 1 \leq i < j < k < l \leq 6 \tag{4}
\]
hold, and there are no more (quadratic) relations between them.

Proof. Note that in addition to nonnegativity: \(f(X,Y) \geq 0\), we have symmetry: \(f(X,Y) = f(Y,X)\), and also zero property: \(f(X,X) = 0\).

By virtue of the last property, the coefficients of \(x_{1}y_{2}\) and \(y_{1}x_{2}\) are opposite in all terms of the representation
\[
f(X,Y) = \sum_i (\alpha_i x_{1}y_{2} + \beta_i y_{1}x_{2} + \ldots)^2,
\]
i.e. \(\beta_i = -\alpha_i\) for all \(i\). Hence
\[
f(X,Y) = \sum_i (\alpha_i z_{1,2} + \gamma_i x_{1}y_{3} + \delta_i y_{1}x_{3} + \ldots)^2
\]
and the procedure can be continued.

As for the relations between the \(z_{i,j}\)-s, assume that there holds a nontrivial quadratic identity \(g(Z) = 0\) containing the term \(z_{1,2}^2\). Then also \(x_{1}^2y_{2}^2\) is present, however, this latter can only occur in \(z_{1,2}^2\), therefore a term \(-z_{1,2}^2\) is needed to cancel it, which contradicts the non-triviality. In a similar way we see that there is no term of type \(z_{1,2} z_{1,3}\) occurring in a non-trivial identity.

Finally, assume we have a non-trivial identity containing the term \(z_{1,2} z_{3,4}\) (Its coefficient can be supposed to be unity.) Then \(x_{1} y_{2} x_{3} y_{4}\) is a part of the (expanded) identity. In contrast to the above cases, this occurs in two additional terms: in \(z_{1,4} z_{2,3}\) and in \(z_{1,3} z_{2,4}\), to produce the non-trivial identity
\[
z_{1,2} z_{3,4} + z_{1,4} z_{2,3} - z_{1,3} z_{2,4} = 0.
\]
Since \(x_{1} y_{2} x_{3} y_{4}\) occurs only in the expansion of the three above terms, there are no more non-trivial identities containing it. \(\square\)

Before proving the theorem, we formulate the standard primal and dual semidefinite programs:
\[
\min \{ C \bullet X : \ X \geq 0, \ A_i \bullet X = b_i, \ 1 \leq i \leq m \} \quad (Primal)
\]
max \{b^T y : S \equiv C - \sum_{i=1}^{m} y_i A_i \geq 0\} \quad (Dual)

where all matrices are \(n\)-th order real symmetric, \(m\) is the number of constraints, \(C\) and \((A_i)_{1}^{m}\) are given, vector \(b\) of length \(m\) is also given, while the primal matrix \(X\) and the dual matrix \(S\) (the so-called ‘slack’ matrix - sometimes denoted by \(Z\)) together with the \(m\)-vector \(y\) are the output of the program, \(\bullet\) denotes the standard scalar product \(A \bullet B = \text{trace}(AB)\) for symmetric matrices and \(\geq\) stands for the Loewner ordering: \(A \geq B\) iff \(A - B\) is positive semidefinite, in short: psd.

Turning to our case, denote by \((A_i)_{1}^{15}\) the constraint matrices corresponding to the basic identities (4) mentioned in the Lemma. Since these are homogeneous equations, the \(b_i\)-s are zero. In an interesting way, both the order \(n = \binom{6}{2}\) and the number of constraints \(m = \binom{6}{1}\) equals 15.

Nevertheless we will need also the identity \(I\) as a constraint matrix to get a sum of squares decomposition, and – to emphasize its speciality – we associate it with index zero, i.e. we write \(A_0 = I\) and get the concrete primal-dual pair of SDP programs:

\[
\begin{align*}
\text{(Primal)} & \quad \min \{ C \bullet X : X \geq 0, \, \text{tr}X = 1, \, A_i \bullet X = 0, \, 1 \leq i \leq 15 \} \\
\text{(Dual)} & \quad \max \{ y_0 : S \equiv C - y_0 I - \sum_{i=1}^{15} y_i A_i \geq 0\}
\end{align*}
\]

After this preparation we can prove our theorem.

**Proof.** To prove Theorem 2, we specify in detail the data for the SDP above and explain the results obtained. Considering the band-width of matrices \(C, X\) and \(S\), a good order of the \(z_{i,j}\)’s is

\[
(\bar{z}_{2,5}, \bar{z}_{3,4}, \bar{z}_{1,2}, \bar{z}_{2,6}, \bar{z}_{1,4}, \bar{z}_{2,3}, \bar{z}_{4,5}, \bar{z}_{4,6}, \bar{z}_{1,3}, \bar{z}_{3,5}, \bar{z}_{2,4}, \bar{z}_{1,5}, \bar{z}_{1,6}, \bar{z}_{3,6}, \bar{z}_{5,6}).
\]

Then, denoting by \(z\) the corresponding column vector, it holds that \(f(X, Y) = z^T C z\) for \(C\) appropriately defined. To this, we describe the common block-structure of the matrices \(C, S, X\). All these matrices are block-diagonal with two \(4 \times 4\) blocks and a \(3 \times 3\) block, while the remaining \(4 \times 4\) block is diagonal. In case of \(C\) e.g. these blocks will be denoted by \(C_4, C_4', C_3\) and \(C_d\). Here, \(C_4'\) is diagonally similar to \(C_4\) through \(\text{diag}(1, 1, 1, -1)\), hence the eigenvalues of \(C_4'\) and \(C_4\) coincide. The whole matrix is

\[
C = C_4 \oplus C_4' \oplus C_3 \oplus C_d,
\]

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and the same direct sum representation holds for the optimal primal and dual matrices $X$ and $S$. As regards $C$, we have

$$C_4 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad C_d = 2I_4.$$

Now we can explain the output of our program. The optimal value of the objective is negative: $y_0 = -\frac{1}{7}$, indicating that (1) is not a SOS, but the modified quartics

$$2\frac{1}{7} ||X||^2||Y||^2 - 2\text{trace}^2(X^TY) - ||XY - YX||^2$$

is a sum of squares, $2\frac{1}{7}$ being the smallest number with this property. The only nonzero $y$-s are $y_1 = y_2 = 5/7$, they correspond to the basic relations (4) with indices $(1, 2, 3, 4)$ and $(2, 3, 4, 5)$.

For the optimal matrix $S$ we have $S = S_4 \oplus S'_4 \oplus S_3 \oplus S_d$ with

$$S_4 = \frac{1}{7} \begin{pmatrix} 15 & -5 & 0 & 0 \\ -5 & 15 & -5 & 0 \\ 0 & -5 & 8 & -7 \\ 0 & 0 & -7 & 8 \end{pmatrix}, \quad S_3 = \frac{1}{7} \begin{pmatrix} 8 & -7 & -2 \\ -7 & 8 & -2 \\ -2 & -2 & 8 \end{pmatrix}, \quad S_d = \frac{15}{7}I_4,$$

which yields the wanted sum of squares decomposition. The optimal primal matrix is $X = X_4 \oplus X'_4 \oplus X_3 \oplus X_d$, where

$$X_4 = \frac{2}{735} \begin{pmatrix} 1 & 3 & 8 & 7 \\ 3 & 9 & 24 & 21 \\ 8 & 24 & 64 & 56 \\ 7 & 21 & 56 & 49 \end{pmatrix}, \quad X_3 = \frac{9}{245} \begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix}, \quad X_d$$

and $X_d$ is the fourth order zero matrix. Using the block-structure, the positive semidefiniteness of $X$ and $S$ and the complementarity condition $XS = 0$ can easily be checked (cf. the Karush-Kuhn-Tucker necessary conditions). Also, strict complementarity holds, in particular $\text{rank}(X) = \text{def}(S) = 3$. \hfill $\square$

Notice that in general – unlike linear programming – rational data for a SDP problem does not necessarily result in rational solution!
4 On a parametric version

To get more insight into the problem, we insert two parameters $\alpha$ and $\beta$ to investigate the SOS representability of the biquadratics

$$2 \sum_{1 \leq i < j \leq 6} z_{i,j}^2 - \alpha(z_{1,3} + z_{2,4} + z_{3,5})^2 - \beta(z_{1,2} + z_{2,6})^2 - \beta(z_{4,5} - z_{4,6})^2. \quad (5)$$

(The reason for the two $\beta$'s is that these terms behave similarly.) It turns out that only the first two constraints $A_1 \cdot X = 0$ and $A_2 \cdot X = 0$ will be active with $y_1 = y_2$, and $y_i = 0$, $i \geq 3$, as in the above special case of $\alpha = \beta = 1$.

This means that our problem reduces to finding the optimal $y_0, y_1$ for a given pair $(\alpha, \beta) \in [0, 1]^2$ such that

$$(2 - y_0)f_0 - \alpha f_1 - \beta f_2 - y_1 F_1 - y_2 F_2 \geq 0 \quad (6)$$

and $y_0$ is maximum, where we use the abbreviations

$$f_0 = \sum_{i<j}^6 z_{i,j}^2, \quad f_1 = (z_{1,3} + z_{2,4} + z_{3,5})^2, \quad f_2 = (z_{1,2} + z_{2,6})^2 + (z_{4,5} - z_{4,6})^2,$$

$$F_1 = 2(z_{1,2} z_{3,4} + z_{1,4} z_{2,3} - z_{1,3} z_{2,4}), \quad F_2 = 2(z_{2,3} z_{4,5} + z_{2,5} z_{3,4} - z_{2,4} z_{3,5})$$

in connection with the notations

$$p = 2 - y_0, \quad q = y_1 = y_2, \quad \text{and} \quad F = F_1 + F_2$$

to write (6) in the simpler form

$$pf_0 - \alpha f_1 - \beta f_2 - qF.$$ 

**Observation.** Assume that for some $(\alpha, \beta) \in [0, 1]^2$ we know the optimal values of $y_0, y_1$, i.e. the optimal $p$ and $q$. Then by multiplying through the coefficient vector $(\alpha, \beta, p, q)$ by $2/p$ we get $(\alpha', \beta', p', q')$ with

$$\alpha' = \frac{2\alpha}{p}, \quad \beta' = \frac{2\beta}{p}, \quad p' = 2, \quad q' = \frac{2q}{p},$$

showing that for this new $(\alpha', \beta')$ we have $y_0' = 0$.

**Example.** Let us calculate the largest $\alpha = \beta$ for which (5) is SOS! (Theorem 2 tells us that this $\alpha < 1$.) For $\alpha = \beta = 1$ we know that $y_0 = -\frac{1}{7}$, thus $p = \frac{15}{7}$, and $q = y_1 = \frac{5}{7}$. The transformed variables are $\alpha' = \beta' = \frac{14}{15}$ and $q' = \frac{1}{3}$. 
Table 1: The unit square: optimal values

<table>
<thead>
<tr>
<th>region</th>
<th>name</th>
<th>$p = 2 - y_0$</th>
<th>$q = y_1$</th>
<th>def(S)</th>
<th>rank(X)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta = 0$</td>
<td></td>
<td>$2 \alpha$</td>
<td>$\alpha$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\beta &lt; 0.8 \alpha$</td>
<td>$R_1$</td>
<td>$2 \alpha$</td>
<td>$\alpha$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\beta = 0.8 \alpha$</td>
<td></td>
<td>$2 \alpha$</td>
<td>$\alpha$</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$\beta \in (0.8\alpha, 1.5\alpha)$</td>
<td>$R_3$</td>
<td>$2 \beta$</td>
<td>$0$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$\beta = 1.5 \alpha$</td>
<td></td>
<td>$2 \beta$</td>
<td>$0$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\beta &gt; 1.5 \alpha$</td>
<td>$R_2$</td>
<td>$2 \beta$</td>
<td>$0$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\alpha = 0$</td>
<td></td>
<td>$2 \beta$</td>
<td>$0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Consequently the polynomial $2f_0 - \frac{14}{15}(f_1 + f_2)$ is not only nonnegative but also SOS (and the point $(\alpha, \beta) = (\frac{14}{15}, \frac{14}{15})$ lies on the border of the ‘good’ and ‘bad’ cases).

In the next theorem we summarize the results obtained for parameters $(\alpha, \beta)$ from the unit square.

**Theorem 3** Table 1 gives the optimal values for the parametrized problem (6).

The optimal $p$ and $q$ for the middle sector $R_3$ are

$$p = -\alpha\beta \frac{2\alpha + 9\beta + \sqrt{(8\alpha - 9\beta)^2 + 15\beta^2}}{4\alpha^2 - 12\alpha\beta + \beta^2} \quad (7)$$

$$q = \beta(6p(60\alpha^2 - 4\alpha\beta - 9\beta^2) + 15\alpha\beta(\beta - 6\alpha)) \quad (8)$$

In addition to this “radial” characterization, the level lines also can be described: these are curves, along which the optimal $y_0$ is constant. The case $y_0 = 0$, i.e. $p = 2$ can be seen on the figure, cf. also (10). (For another instance, the dotted line starting at $\alpha = 0$, $\beta = 0.5$ and ending at $\alpha = 0.5$, $\beta = 0$ belongs to $p = 1$.) The indices of region names correspond to the defects of the optimal dual matrix $S$. The right upper darkened region within $R_3$ refers to points $(\alpha, \beta)$ for which (5), i.e. $2f_0 - \alpha f_1 - \beta f_2$ is not a sum of squares.

**Proof.** The block-structure of the special case $\alpha = \beta = 1$ keeps on holding,
and the blocks at issue depend just on one of the parameters $\alpha$ and $\beta$:

$$S_3(\alpha) = \begin{pmatrix} p - \alpha & -\alpha & q - \alpha \\ -\alpha & p - \alpha & q - \alpha \\ q - \alpha & q - \alpha & p - \alpha \end{pmatrix}, \quad S_4(\beta) = \begin{pmatrix} p & -q & 0 & 0 \\ -q & p & -q & 0 \\ 0 & -q & p - \beta & -\beta \\ 0 & 0 & -\beta & p - \beta \end{pmatrix},$$

as it easily follows by considering the polynomial (6).

Let us begin with the case $\beta = 0$. Then $S_4$ (and also $S_4'$) is psd, and our 'work matrix' is $S_3$. Its $(1,1)$ entry, $p - \alpha \geq 0$, and the nonnegativity of the left upper 2-minor implies $|p - \alpha| \geq |\alpha|$. Since both sides are nonnegative, it follows that $p \geq 2\alpha$. On the other hand, the determinant of $S_3$ equals

$$|S_3| \equiv \det(S_3) = p \{p^2 - 2q^2 - \alpha(3p - 4q)\},$$

which must vanish at the optimal variables, therefore

$$\left(p - \frac{3}{2} \alpha\right)^2 = 2(q - \alpha)^2 + \frac{1}{4} \alpha^2.$$ 

From this we get $|p - \frac{3}{2} \alpha| \geq \frac{\sqrt{2}}{2}$, and, since this holds without absolute value as well, we conclude that the optimal variables are $q = 0$ and $p = 2\alpha$ (note that maximizing $y_0$ is equivalent to minimizing $p = 2 - y_0$).
All this holds for $R_1$, i.e. for $\beta$ “small” – until we arrive at $|S_4(\beta)| = 0$. Before that moment we still have $p = 2\alpha$, $q = \alpha$ and

$$S_3 = \alpha \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

giving one rank decrease. As regards $S_4$, for $\beta \leq (4/5)\alpha$ we have

$$S_4 \geq \begin{pmatrix} p & -\alpha & 0 & 0 \\ -\alpha & p & -\alpha & 0 \\ 0 & -\alpha & p - \frac{4}{5} \alpha & -\frac{4}{5} \alpha \\ 0 & 0 & -\frac{4}{5} \alpha & p - \frac{4}{5} \alpha \end{pmatrix} = \alpha \begin{pmatrix} 10 & -5 & 0 & 0 \\ -5 & 10 & -5 & 0 \\ 0 & -5 & 6 & -4 \\ 0 & 0 & 4 & 6 \end{pmatrix}$$

in the sense of the semidefinite (Loewner) ordering, which easily follows by (6). Matrix $S_4$ becomes active, if equality is attained: $\beta = (4/5)\alpha$. Then $S_4$ becomes a psd matrix with rank 3, therefore $S_4$ and $S_4'$ yield two further rank losses. (The situation on the border of $R_1$ and $R_3$ can be understood by thinking of the continuity of the roots depending on the parameters.)

The cases $\alpha = 0$ and $\beta > 1.5 \alpha$ are similar, hence we give only the necessary formulas. The determinant of $S_4$ is

$$|S_4| = p \{p^2(p - 2\beta) - q^2(2p - 3\beta)\}.$$ 

Inequality $p \geq 2\beta$ follows similarly to the case $p \geq 2\alpha$. On the other hand, $|S_4| = 0$ can be rewritten as

$$p(p - \beta)^2 + 3q^2\beta = p(2q^2 + \beta^2),$$

whence we conclude $(p - \beta)^2 \leq 2q^2 + \beta^2$, i.e. $p(p - 2\beta) \leq 2q^2$, and it follows that the optimal values ($p \to \min!$) are $q = 0$ and $p = 2\beta$. The matrix

$$S_4 = \beta \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

is psd with defect one, giving (together with $S_4'$) rank loss=2 for the whole matrix $S$. As regards $S_3$, for $(\alpha, \beta) \in R_2$ it equals

$$S_3(\alpha) = pI_3 - \alpha ee^T = \alpha \left( \frac{p}{\alpha} I_3 - ee^T \right), \quad e = (1, 1, 1)^T,$$
which is positive definite for \( p/\alpha > 3 \), i.e. for \( \beta > (3/2)\alpha \), and psd (with rank=2) if equality holds:

\[
S_3 = \alpha \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.
\]

In this latter case, i.e. if \( \beta = (3/2)\alpha \), the defect of \( S \) becomes 3, since not only \( S_4 \) and \( S'_4 \), but also \( S_3 \) yields a rank loss.

The middle region \( R_3 \) can be characterized by that the SDP program annihilates both determinants. The equations \(|S_3(\alpha)| = 0 \) and \(|S_4(\beta)| = 0 \) are equivalent to

\[
p^2 - 2q^2 = \alpha(3p - 4q) \quad \text{and} \quad p(p^2 - 2q^2) = \beta(2p^2 - 3q^2),
\]

from which we can express \( p \) and \( q \) by means of \( \alpha \) and \( \beta \) to get (7) and (8), and also \( \beta \) by help of \( \alpha \) and \( p \):

\[
\varphi_p(\alpha) = \frac{2p}{3} - \frac{2p^3}{3}(p^2 + 9\alpha p - 12\alpha^2 + 6\alpha\sqrt{2(p - \alpha)(p - 2\alpha)})^{-1}
\]

In the important special case \( p = 2 \) we have the function \( \varphi : \left[\frac{2}{3}, 1\right] \to \left[\frac{4}{5}, 1\right] \) defined by

\[
\varphi(\alpha) \equiv \varphi_2(\alpha) = \frac{4}{3} - \frac{5}{3}(2 + 9\alpha - 6\alpha^2 + 6\alpha\sqrt{1 - \alpha})(2 - \alpha))^{-1}.
\]

The critical values of \( \varphi \) are:

\[
\varphi\left(\frac{2}{3}\right) = 1, \quad \varphi'\left(\frac{2}{3}\right) = 0, \quad \varphi(1) = \frac{4}{5}, \quad \varphi'(1) = -\infty.
\]

The set of points in the unit square, for which (5) is a SOS are delimited by the \( \alpha \) and \( \beta \) axes, the horizontal line segment \((0, 1)\) to \((\frac{2}{3}, 1)\), the graph of \( \varphi \) and the vertical line segment \((1, 0)\) to \((1, \frac{4}{5})\). All other curves with different \( p \) (e.g. that with \( p = 1 \) dotted on the figure) are proportional to this one, since equations (9) are homogeneous.

\[\square\]

**Remark 3** Consider the ellipse with center in \((\frac{2}{3}, \frac{4}{5})\) and vertices \((1, \frac{4}{5})\), \((\frac{2}{3}, 1)\), the right upper quarter of which is close to the graph of \( \varphi \). The elementary equality \((\frac{14}{15} - \frac{3}{5})^2 + (1 - \frac{4}{5})^2 = (1 - \frac{2}{3})^2\) shows that the projection of the ‘border point’ \((\frac{14}{15}, \frac{12}{15})\) onto the longer axis of the ellipse is just its focus \((\frac{14}{15}, \frac{2}{5})\)!

As for their measures, the approximate area of the ‘bad’ (shadowed) region is 0.0121, while that above the ellipse amounts to \(\frac{1}{15} - \frac{\pi}{60} \approx 0.0143\).
Remark 4 For the interested reader we give some ‘nice’ rational solutions: in addition to the known quadruples \((\alpha, \beta, p, q) = (1, 1, \frac{15}{7}, \frac{2}{3})\) and \((\frac{14}{15}, \frac{14}{15}, 2, \frac{2}{3})\) we have e.g. \((\alpha, \beta, p, q) = (\frac{34}{35}, \frac{17}{15}, 2, \frac{4}{3}), (\frac{17}{31}, \frac{68}{23}, 6, 1)\), or \((\frac{31}{16}, \frac{124}{50}, 5, \frac{5}{8})\).

Considering the derivatives also is of interest: the slope of \(\varphi_p\) for \(p = 2\frac{1}{7}\) at \(\alpha = 1\) equals \(-\frac{3}{4}\), which can be used to define a new problem with the same solution! Replace to this the identity \(A_0 = I\) by two matrices \(A_{16}\) and \(A_{17}\) associated with the quadratic forms \(f_1\) and \(f_2\), and set the corresponding coordinates of \(b\) equal to 3 and 4 (coming from the numerator and denominator of the ratio \(-\frac{3}{4}\) above). The result will coincide with that of Theorem 2.

Remark 5 In [6] the authors write: “Unfortunately, the nature of a parametric SDP is far more complicated [than LP] due to regions of nonlinearity of \(\varphi(\gamma)\).” (The function \(\phi(\gamma) = C(\gamma) \bullet X(\gamma)\) is the primal objective depending on the parameter.) In light of this, present problem seems to be a refreshing exception: the nonlinearity (cf. the functions \(\varphi_p\)) can be handled by means of elementary functions.

References


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