Multiplying balancing numbers

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Abstract. In this paper we prove some results on multiplying balancing and cobalancing numbers and (k, l)-power numerical centers.

1 Introduction

The sequence $R = \{R_i\}_{i=0}^{\infty} = R(A, B, R_0, R_1)$ is called a second order linear recurrence sequence if the recurrence relation

$$R_i = AR_{i-1} + BR_{i-2} \ (i \geq 2)$$

holds, where $A, B \neq 0, R_0, R_1$ are fixed rational integers and $|R_0| + |R_1| > 0$. A positive integer n is called a balancing number (see [3] and [5]) if

$$1 + \cdots + (n - 1) = (n + 1) + \cdots + (n + r)$$

holds for some positive integer r. The sequence of balancing numbers is denoted by B_m $(m = 1, 2, \dots)$. As one can easily check, we have $B_1 = 6$ and $B_2 = 35$. Note that by a result of Behera and Panda [3], we have

$$B_{m+1} = 6B_m - B_{m-1} \qquad (m > 1).$$

In that paper they proved that, there are infinitely many balancing numbers.

In [7] K. Liptai searched for those balancing numbers which are Fibonacci numbers, too. Using the results of A. Baker and G. Wüstholz [2] he proved

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that there are no Fibonacci balancing numbers. Similarly in [8] he proved that there are no Lucas balancing numbers. Using an other method L. Szalay [12] got the same result.

In [9] Liptai, Luca, Pintér and Szalay generalized the concept of balancing numbers in the following way. Let y, k, l be fixed positive integers with $y \geq 4$. A positive integer x with $x \leq y - 2$ is called a (k, l)-power numerical center for y if

$$1^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l.$$

In [9] several effective and ineffective finiteness results were proved for (k, l)-power numerical centers.

Later G.K. Panda and P.K. Ray (see [10]) slightly modified the definition of balancing number and introduced the notion of cobalancing number. A positive integer n is called a *cobalancing number* if

$$1 + 2 + \cdots + (n-1) + n = (n+1) + (n+2) + \cdots + (n+K)$$

for some $K \in \mathbb{N}$. In this case K is called the cobalancer of n.

They also proved that the cobalancing numbers fulfill the following recurrence relation

$$b_{n+1} = 6b_n - b_{n-1} + 2 \quad (n > 1),$$

where $b_0 = 1$ and $b_1 = 6$. Moreover they found that every balancer is a cobalancing number and every cobalancer is a balancing number.

In [11] G. K. Panda gave another possible generalization of balancing numbers. Let $\{a_m\}_{m=0}^{\infty}$ be a sequence of real numbers. We call an element a_n of this sequence a *sequence-balancing* number if

$$a_1 + a_2 + \cdots + a_{n-1} = a_{n+1} + a_{n+2} + \cdots + a_{n+k}$$

for some $k \in \mathbb{N}$. Similarly, one can define the notion of *sequence cobalancing numbers*. In [11] it was proved that there does not exist any sequence balancing number in the Fibonacci sequence.

As a generalization of the notion of a balancing number A. Bérczes, K. Liptai and I. Pink call a binary recurrence $R = R(A, B, R_0, R_1)$ a *balancing sequence* if

$$R_1 + R_2 + \ldots + R_{n-1} = R_{n+1} + R_{n+2} + \ldots + R_{n+k}$$

holds for some $k \geq 1$ and $n \geq 2$.

In [4] they proved that that any sequence $R = R(A, B, 0, R_1)$ with the condition $D = A^2 + 4B > 0$, $(A, B) \neq (0, 1)$ is not a balancing sequence.

T. Kovács, K. Liptai and P. Olajos in [6] extended the concept of balancing numbers to arithmetic progressions. Let $a > 0$ and $b \geq 0$ be coprime integers. If for some positive integers n and r we have

$$(a+b)+\cdots+(a(n-1)+b) = (a(n+1)+b)+\cdots+(a(n+r)+b)$$

then we say that $an+b$ is an (a,b)-balancing number. They proved several effective finiteness and explicit results about them. In the proofs they combined the Baker's method, the modular method developed by Wiles and others, the Chabauty method and the theory of elliptic curves.

In this paper we study a further generalization of balancing numbers. The idea is due to A. Behera and G. K. Panda. A positive integer n is called a *multiplying balancing number* if

$$1\cdot 2\cdots(n-1) = (n+1)(n+2)\cdots(n+r) \tag{1}$$

for some positive integer r. The number r is called the *balancer* corresponding to the multiplying balancing number n. The cobalancing numbers have a similar definition. A positive integer n is called a *multiplying cobalancing number* if

$$1\cdot 2\cdots(n-1)n = (n+1)(n+2)\cdots(n+r) \tag{2}$$

for some positive integer r. The number r is called the *cobalancer* corresponding to the multiplying cobalancing number n.

Using the concept of K. Liptai, F. Luca, . Pintér and L. Szalay ([9] we can get further generalization. Let m, k, l be fixed positive integers with $m \geq 4$. A positive integer n with $n \leq m-2$ is called a (k,l)*-power multiplying balancing number for* m if

$$1^k\cdots(n-1)^k = (n+1)^l\cdots(m-1)^l. \tag{3}$$

2 The results

Throughout the paper let p the greatest odd prime, which is less than the multiplying balancing number n, where $n \geq 4$. In the first theorem we prove that only one multiplying balancing number exists.

Theorem 1 *The only multiplying balancing number is* $n = 7$ *with the balancer* $r = 3$.

In the proof we use 4 lemmas.

Lemma 1 *There is no prime among the factors of the right side of the equation*

$$1 \cdot 2 \cdots (n-1) = (n+1)(n+2)\cdots(n+r)$$

Proof. Suppose that z is a prime among the factors of the right side. It is clear that z is not in the prime decomposition of the left side of the equation (1). Hence the prime decomposition of the right side is not the same as the left's. Thus the lemma is proved. $\square$

Let us use the function $\alpha_2 \colon \mathbb{N} \to \mathbb{N}$, $\alpha_2(x) := \sum_{k=1}^{[\log_2 x]} \left[\frac{x}{2^k}\right]$, where $x \geq 2$ and $\alpha_2(x)$ shows the index of the prime 2 in $x!$.

Lemma 2 $x - \log_2 x - 2 < \alpha_2(x) < x$

Proof.

$$\alpha_2(x) = \left[\frac{x}{2^1}\right] + \left[\frac{x}{2^2}\right] + \left[\frac{x}{2^3}\right] + \cdots + \left[\frac{x}{2^k}\right] \leq \frac{x}{2^1} + \frac{x}{2^2} + \frac{x}{2^3} + \cdots + \frac{x}{2^k} =$$

$$= x\left(\frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^k}\right) = x\left(1 - \frac{1}{2^k}\right) \leq x - 1 < x$$

$$\alpha_2(x) > \underbrace{\left(\frac{x}{2^1} - 1\right) + \left(\frac{x}{2^2} - 1\right) + \cdots + \left(\frac{x}{2^k} - 1\right)}_{[\log_2 x]} =$$

$$= x\left(1 - \frac{1}{2^k}\right) - [\log_2 x] = x - \frac{x}{2^k} - [\log_2 x] > x - \log_2 x - 2$$

$\square$

Lemma 3 *If n is multiplying balancing number and r is the balancer, furthermore $n > 64$ then*

$$\frac{3(n+1)}{2} < n + r.$$

Proof. From (1) it follows, that

$$(n-1)! \cdot (n)! = (n+r)!.$$

If (1) is true then

$$\alpha_2(n-1) + \alpha_2(n) = \alpha_2(n+r)$$

Using Lemma 2 we get

$$n - 2\log_2 n - 5 < r$$

We can replace $\log_2 n$ with $\frac{n}{8}$ if $n > 64$, that is

$$2n - \frac{2n}{8} - 5 < n + r$$

If $n > 64$ we get

$$\frac{3(n+1)}{2} < \frac{3(n+1)}{2} + \frac{2n}{8} - 6.5 < n + r.$$

□

Proof.[Proof of Theorem 1] Using a results of M. El Bachraoui ([1]) we get that, if $n \geq 2$ then exists a p prime satisfying the inequality

$$n < p < \frac{3(n+1)}{2}.$$

Hence the right side of (1) contains a prime if $n > 64$. But Lemma 1 says there is no prime on the right side of the equation. The conclusion is that, if $n > 64$ there is no multiplying balancing numbers. It can be checked easily if $n = 2, \dots, 64$ then there is only one number satisfying the equation (1). We get $n = 7$, that is the theorem is proved. □

Theorem 2 *There is no multiplying cobalancing number.*

In the proof we use the following lemma.

Lemma 4 *Using our notation the following inequalities are true*

$$p < n < 2p \leq n + r < 3p.$$

Proof. Suppose that $n \geq 2p$. The interval $[p, 2p]$ always contains a prime, so there is a prime greater than p and lower than n which is impossible because of the definition of p. Hence $n < 2p$. On the left side of the equation (1) the index of p is 1, consequently on the right side the index of p is also 1 in the prime decomposition. So we can write the following inequalities $2p \leq n + r < 3p$. □

Proof. [Proof of Theorem 2] Using a result of Csebisev we get that there is a prime z between p and $2p$. Because of Lemma 4 we have to analyse three cases $z = n$, $z > n$ and $z < n$. If $z > n$ then the prime decomposition of the left and right side is not the same. Now let $z < n$.This situation contradicts the fact that p is the greatest odd prime which is less than n. The last case is $z = n$. Hence $n + r \geq 2z$ because of the prime factor z. Thus the left side of the equation (2) has as many factor as the right side has which is obviously impossible. First and last there is no cobalancing numbers. □

The following theorem deals with the (k, l)-power numerical centers.

Theorem 3 *If* $n \geq 4$ *then there is only one* (k, l)*-power numerical centers. The only solution is* $n = 7$*,* $m = 11$ *and* $k = l$*.*

Proof. First we prove that if n is a (k, l)-power numerical center for m then $k = l$. Using Lemma 4 the index of p in the equation (3) is k on the left side and l on the right side in the prime decomposition. The index of p have to be equal on the left and right side. So $k = l$.

So we get that n satisfies (1) if and only if n satisfies (3). So if $n \geq 4$ there is only one (k, l)-power numerical center. It is $n = 7$, $m = 11$ and $l = k$. □

Remark 1 *If* $p = 2$ *and* $n = 3$ *we get the equation*

$$1^k \cdot 2^k = 4^l.$$

In this case $n = 3$ (k, l)*-power numerical center for* $m - 5$ *and there are infinitely many* (k, l) *pairs with* $k = 2l$*.*

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