Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz function

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Abstract. In the present paper we introduce some sequence spaces combining lacunary sequence, invariant means in 2-normed spaces defined by Musielak-Orlicz function $\mathcal{M} = (M_k)$. We study some topological properties and also prove some inclusion results between these spaces.

1 Introduction and preliminaries

The concept of 2-normed space was initially introduced by Gahler [2] as an interesting linear generalization of a normed linear space which was subsequently studied by many others see ([3], [9]). Recently a lot of activities have started to study sumability, sequence spaces and related topics in these linear spaces see ([4], [10]).

Let $X$ be a real vector space of dimension $d$, where $2 \leq d < \infty$. A 2-norm on $X$ is a function $\|.,.\| : X \times X \rightarrow \mathbb{R}$ which satisfies

(i) $\|x, y\| = 0$ if and only if $x$ and $y$ are linearly dependent
(ii) $\|x, y\| = \|y, x\|$
(iii) $\|\alpha x, y\| = |\alpha\|\|x, y\|$, $\alpha \in \mathbb{R}$
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all $x, y, z \in X$.

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The pair \((X, \|\cdot\|)\) is then called a 2-normed space see [3]. For example, we may take \(X = \mathbb{R}^2\) equipped with the 2-norm defined as \(\|x, y\| = \text{the area of the parallelogram spanned by the vectors } x \text{ and } y\) which may be given explicitly by the formula

\[
\|x_1, x_2\|_E = \text{abs} \left( \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \right).
\]

Then, clearly \((X, \|\cdot\|)\) is a 2-normed space. Recall that \((X, \|\cdot\|)\) is a 2-Banach space if every cauchy sequence in \(X\) is convergent to some \(x\) in \(X\).

Let \(\sigma\) be the mapping of the set of positive integers into itself. A continuous linear functional \(\varphi\) on \(l_\infty\), is said to be an invariant mean or \(\sigma\)-mean if and only if

(i) \(\varphi(x) \geq 0\) when the sequence \(x = (x_k)\) has \(x_k \geq 0\) for all \(k\),

(ii) \(\varphi(e) = 1\), where \(e = (1, 1, 1, \ldots)\) and

(iii) \(\varphi(x_{\sigma(k)}) = \varphi(x)\) for all \(x \in l_\infty\).

If \(x = (x_n)\), write \(Tx = T_{x_n} = (x_{\sigma(n)})\). It can be shown in [11] that

\[V_\sigma = \{x \in l_\infty \mid \lim_{k} t_{kn}(x) = 1, \text{ uniformly in } n, 1 = \sigma - \lim x\},\]

where

\[t_{kn}(x) = \frac{x_n + x_{\sigma^1n} + \ldots + x_{\sigma^kn}}{k + 1}.
\]

In the case \(\sigma\) is the translation mapping \(n \to n + 1\), \(\sigma\)-mean is often called a Banach limit and \(V_\sigma\), the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences see [6].

By a lacunary sequence \(\theta = (k_r)\) where \(k_0 = 0\), we shall mean an increasing sequence of non-negative integers with \(k_r - k_{r-1} \to \infty\) as \(r \to \infty\). The intervals determined by \(\theta\) will be denoted by \(I_r = (k_{r-1}, k_r]\). We write \(h_r = k_r - k_{r-1}\). The ratio \(\frac{k_r}{k_{r-1}}\) will be denoted by \(q_r\). The space of lacunary strongly convergent sequence was defined in [1].

Let \(X\) be a linear metric space. A function \(p : X \to \mathbb{R}\) is called paranorm, if

(i) \(p(x) \geq 0\), for all \(x \in X\)

(ii) \(p(-x) = p(x)\), for all \(x \in X\)

(iii) \(p(x + y) \leq p(x) + p(y)\), for all \(x, y \in X\)

(iv) if \((\sigma_n)\) is a sequence of scalars with \(\sigma_n \to \sigma\) as \(n \to \infty\) and \((x_n)\) is a sequence of vectors with \(p(x_n - x) \to 0\) as \(n \to \infty\), then \(p(\sigma_n x_n - \sigma x) \to 0\) as \(n \to \infty\).
Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz function

A paranorm $p$ for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair $(X, p)$ is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [12], Theorem 10.4.2, P-183).

An orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to define the following sequence space. Let $w$ be the space of all real or complex sequences $x = (x_k)$, then

$$l_M = \left\{ x \in w \mid \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called a Orlicz sequence space. Also $l_M$ is a Banach space with the norm

$$||x|| = \inf\left\{ \rho > 0 \mid \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Also, it was shown in [5] that every Orlicz sequence space $l_M$ contains a subspace isomorphic to $l_p$ ($p \geq 1$). The $\Delta_2$- condition is equivalent to $M(Lx) \leq LM(x)$, for all $L$ with $0 < L < 1$. An Orlicz function $M$ can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt$$

where $\eta$ is known as the kernel of $M$, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, $\eta$ is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $M = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see ([7], [8]). A sequence $N = (N_k)$ is called a complementary function of a Musielak-Orlicz function $M$

$$N_k(v) = \sup \{|v|u - M_k | u \geq 0\}, \text{ for some } c > 0, k = 1, 2, \ldots$$

For a given Musielak-Orlicz function $M$, the Musielak-Orlicz sequence space $t_M$ and its subspace $h_M$ are defined as follows

$$t_M = \left\{ x \in w \mid I_M(cx) < \infty, \text{ for some } c > 0 \right\},$$
$$h_M = \left\{ x \in w \mid I_M(cx) < \infty, \text{ for all } c > 0 \right\}.$$
where $I_M$ is a convex modular defined by

$$I_M(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_M.$$ 

We consider $t_M$ equipped with the Luxemburg norm

$$\|x\| = \inf \left\{ k > 0 \mid I_M \left( \frac{x}{k} \right) \leq 1 \right\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf \left\{ \frac{1}{k} \left( 1 + I_M(kx) \right) \mid k > 0 \right\}.$$ 

Let $M = (M_k)$ be a Musielak-Orlicz function, $(X, \|\cdot\|)$ be a 2-normed space and $p = (P_k)$ be any sequence of strictly positive real numbers. By $S(2-X)$ we denote the space of all sequences defined over $(X, \|\cdot\|)$. We now define the following sequence spaces:

$$w_0^\sigma[M, p, \|\cdot\|]_\theta = \left\{ x \in S(2-X) \mid \lim_{r \to \infty} \frac{1}{H_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{t_{kn}(x)}{\rho} , z \right\| \right) \right]^{p_k} = 0, \right. \rho > 0, \text{ uniformly in } n \right\},$$

$$w_\sigma[M, p, \|\cdot\|]_\theta = \left\{ x \in S(2-X) \mid \lim_{r \to \infty} \frac{1}{H_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{t_{kn}(x-1)}{\rho} , z \right\| \right) \right]^{p_k} = 0, \right. \rho > 0, \text{ uniformly in } n \right\},$$

$$w_\sigma^\infty[M, p, \|\cdot\|]_\theta = \left\{ x \in S(2-X) \mid \sup_{r,n} \frac{1}{H_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{t_{kn}(x)}{\rho} , z \right\| \right) \right]^{p_k} < \infty, \right. \rho > 0, \text{ uniformly in } n \right\},$$

for some $\rho > 0$. When $M(x) = x$ for all $k$, the spaces $w_0^\sigma[M, p, \|\cdot\|]_\theta$, $w_\sigma[M, p, \|\cdot\|]_\theta$, and $w_\sigma^\infty[M, p, \|\cdot\|]_\theta$ reduces to the spaces $w_0^\sigma[p, \|\cdot\|]_\theta$, $w_\sigma[p, \|\cdot\|]_\theta$, and $w_\sigma^\infty[p, \|\cdot\|]_\theta$ respectively.

If $P_k = 1$ for all $k$, the spaces $w_0^\sigma[M, p, \|\cdot\|]_\theta$, $w_\sigma[M, p, \|\cdot\|]_\theta$, and $w_\sigma^\infty[M, p, \|\cdot\|]_\theta$ reduces to $w_0^\sigma[M, \|\cdot\|]_\theta$, $w_\sigma[M, \|\cdot\|]_\theta$, and $w_\sigma^\infty[M, \|\cdot\|]_\theta$ respectively.
The following inequality will be used throughout the paper. If \( 0 \leq p_k \leq \sup p_k = H, \ K = \max(1, 2^{H-1}) \) then
\[
|a_k + b_k|^{p_k} \leq K\{ |a_k|^{p_k} + |b_k|^{p_k} \} \tag{1}
\]
for all \( k \) and \( a_k, b_k \in \mathbb{C} \). Also \( |a|^{p_k} \leq \max(1, |a|^H) \) for all \( a \in \mathbb{C} \).

In the present paper we study some topological properties of the above sequence spaces.

2 Main results

**Theorem 1** Let \( \mathcal{M} = (M_k) \) be Musielak-Orlicz function, \( p = (p_k) \) be a bounded sequence of positive real numbers, then the classes of sequences \( w_0^{\sigma}[\mathcal{M}, p, ||.,||]_0, w_0^\infty[\mathcal{M}, p, ||.,||]_0 \) and \( w_0^\infty[\mathcal{M}, p, ||.,||]_0 \) are linear spaces over the field of complex numbers.

**Proof.** Let \( x, y \in w_0^\infty[\mathcal{M}, p, ||.,||]_0 \) and \( \alpha, \beta \in \mathbb{C} \). In order to prove the result we need to find some \( \rho_3 \) such that
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{t_kn(\alpha x + \beta y)}{\rho_3}, z \right\| \right) \right]^{p_k} = 0, \text{ uniformly in } n.
\]
Since \( x, y \in w_0^\infty[\mathcal{M}, p, ||.,||]_0 \), there exist positive \( \rho_1, \rho_2 \) such that
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{t_kn(x)}{\rho_1}, z \right\| \right) \right]^{p_k} = 0, \text{ uniformly in } n
\]
and
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{t_kn(y)}{\rho_2}, z \right\| \right) \right]^{p_k} = 0, \text{ uniformly in } n.
\]
Define \( \rho_3 = \max(2/|\alpha|\rho_1, 2|\beta|\rho_2) \). Since \( (M_k) \) is non-decreasing and convex
\[
\frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{t_kn(\alpha x + \beta y)}{\rho_3}, z \right\| \right) \right]^{p_k} \leq \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{t_kn(\alpha x)}{\rho_3}, z \right\| + \left\| \frac{t_kn(\beta y)}{\rho_3}, z \right\| \right) \right]^{p_k}
\]
\[
\leq \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{t_kn(x)}{\rho_1}, z \right\| + \left\| \frac{t_kn(y)}{\rho_2}, z \right\| \right) \right] +
\]
\[
+ k \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{t_kn(y)}{\rho_2}, z \right\| \right) \right]
\]
\[
\to 0 \text{ as } r \to \infty, \text{ uniformly in } n.
\]
So that $\alpha x + \beta y \in w_0^\alpha [\mathcal{M}, p, ||\cdot||_\sigma]$. This completes the proof. Similarly, we can prove that $w_\sigma [\mathcal{M}, p, ||\cdot||_\sigma]$ and $w_\sigma^\infty [\mathcal{M}, p, ||\cdot||_\sigma]$ are linear spaces. \hfill \Box

**Theorem 2** Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers. Then $w_0^\alpha [\mathcal{M}, p, ||\cdot||_\sigma]$ is a topological linear spaces paranormed by

$$g(x) = \inf \left\{ \rho \frac{\pi}{\sqrt{H}} : \left( \frac{1}{r^H} \sum_{k \in I_r} \left( M_k \left( \left\| t_k n(x) \rho - z \right\| \right)^{p_k} \right)^{1/H} \right)^{1/\pi} \leq 1, r = 1, 2, \ldots, n = 1, 2, \ldots \right\},$$

where $H = \max(1, \sup_k p_k < \infty)$.

**Proof.** Clearly $g(x) \geq 0$ for $x = (x_k) \in w_0^\alpha [\mathcal{M}, p, ||\cdot||_\sigma]$. Since $M_k(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then

$$\inf \left\{ \rho \frac{\pi}{\sqrt{H}} : \left( \frac{1}{r^H} \sum_{k \in I_r} \left( M_k \left( \left\| t_k n(x) \rho - z \right\| \right)^{p_k} \right)^{1/H} \right)^{1/\pi} \leq 1, r \geq 1, n \geq 1 \right\} = 0.$$

This implies that for a given $\epsilon > 0$, there exists some $\rho_{\epsilon}(0 < \rho_{\epsilon} < \epsilon)$ such that

$$\left( \frac{1}{r^H} \sum_{k \in I_r} \left( M_k \left( \left\| t_k n(x) \rho_{\epsilon} - z \right\| \right)^{p_k} \right)^{1/H} \right)^{1/\pi} \leq 1.$$

Thus

$$\left( \frac{1}{r^H} \sum_{k \in I_r} \left( M_k \left( \left\| t_k n(x) \epsilon\rho_{\epsilon} - z \right\| \right)^{p_k} \right)^{1/H} \right)^{1/\pi} \leq \left( \frac{1}{r^H} \sum_{k \in I_r} \left( M_k \left( \left\| t_k n(x) \rho_{\epsilon} - z \right\| \right)^{p_k} \right)^{1/H} \right)^{1/\pi} \leq 1,$$

for each $r$ and $n$. Suppose that $x_k \neq 0$ for each $k \in \mathbb{N}$. This implies that $t_k n(x) \neq 0$, for each $k, n \in \mathbb{N}$. Let $\epsilon \to 0$, then $\left\| t_k n(x) \epsilon\rho_{\epsilon} - z \right\| \to \infty$. It follows that

$$\left( \frac{1}{r^H} \sum_{k \in I_r} \left( M_k \left( \left\| t_k n(x) \rho_{\epsilon} - z \right\| \right)^{p_k} \right)^{1/H} \right)^{1/\pi} \to \infty$$

which is a contradiction.
Therefore, \( t_{kn}(x) = 0 \) for each \( k \) and thus \( x_k = 0 \) for each \( k \in \mathbb{N} \). Let \( \rho_1 > 0 \) and \( \rho_2 > 0 \) be such that

\[
\left( \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{t_{kn}(x)}{\rho_1}, z \right\| \right) \right)^{\frac{1}{p_k}} \leq 1
\]

and

\[
\left( \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{t_{kn}(y)}{\rho_2}, z \right\| \right) \right)^{\frac{1}{p_k}} \leq 1
\]

for each \( r \). Let \( \rho = \rho_1 + \rho_2 \). Then, we have

\[
\left( \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{t_{kn}(x+y)}{\rho}, z \right\| \right) \right)^{\frac{1}{p_k}} \\
\leq \left( \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{t_{kn}(x)}{\rho_1}, z \right\| \right) \right)^{\frac{1}{p_k}} \\
\leq \left( \frac{1}{h_r} \sum_{k \in I_r} \left[ \frac{\rho_1}{\rho_1 + \rho_2} M_k \left( \left\| \frac{t_{kn}(x)}{\rho_1}, z \right\| \right) \right] \right)^{\frac{1}{p_k}} + \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) \left( \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{t_{kn}(y)}{\rho_2}, z \right\| \right) \right)^{\frac{1}{p_k}} \leq 1.
\]

(by Minkowski’s inequality)
Since $\rho$'s are non-negative, so we have

\[
g(x + y) = \inf \left\{ \rho^\frac{\nu}{\pi} | \left( \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{t_{kn}(x) + t_{kn}(y)}{\rho}, z \right\| \right)^p \right)^{\frac{1}{\pi}} \leq 1, r \geq 1, n \geq 1 \right\}
\]

\[
\leq \inf \left\{ \rho^\frac{\nu}{\pi} | \left( \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{t_{kn}(x)}{\rho}, z \right\| \right)^p \right)^{\frac{1}{\pi}} \leq 1, r \geq 1, n \geq 1 \right\} + \\
+ \inf \left\{ \rho^\frac{\nu}{\pi} | \left( \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{t_{kn}(x)}{\rho}, z \right\| \right)^p \right)^{\frac{1}{\pi}} \leq 1, r \geq 1, n \geq 1 \right\}.
\]

Therefore,

\[
g(x + y) \leq g(x) + g(y).
\]

Finally, we prove that the scalar multiplication is continuous. Let $\lambda$ be any complex number. By definition,

\[
g(\lambda x) = \inf \left\{ \rho^\frac{\nu}{\pi} | \left( \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{t_{kn}(\lambda x)}{\rho}, z \right\| \right)^p \right)^{\frac{1}{\pi}} \leq 1, r \geq 1, n \geq 1 \right\}.
\]

Then

\[
g(\lambda x) = \inf \left\{ |\lambda|^{\frac{\nu}{\pi}} | \left( \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{t_{kn}(x)}{t}, z \right\| \right)^p \right)^{\frac{1}{\pi}} \leq 1, r \geq 1, n \geq 1 \right\}.
\]

where $t = \frac{\rho}{|\lambda|}$. Since $|\lambda|^p \leq \max(1, |\lambda|^{\sup p})$, we have

\[
g(\lambda x) \leq \max(1, |\lambda|^{\sup p}) \inf \left\{ t^\frac{\nu}{\pi} | \left( \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \left\| \frac{t_{kn}(x)}{t}, z \right\| \right)^p \right)^{\frac{1}{\pi}} \leq 1, r \geq 1, n \geq 1 \right\}.
\]

So, the fact that scalar multiplication is continuous follows from the above inequality.

This completes the proof of the theorem.
Theorem 3 Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function. If 
$$\sup_k [M_k(t)]p_k < \infty \text{ for all } t > 0,$$
then 
$$w_\sigma [\mathcal{M}, p, ||.,||_\theta] \subset w_\sigma^\infty [\mathcal{M}, p, ||.,||_\theta].$$

Proof. Let $x \in w_\sigma [\mathcal{M}, p, ||.,||]_\theta$. By using inequality (1), we have
$$\frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{t_{kn}(x)}{\rho}, z \right\| \right) \right]^{p_k} \leq \frac{K}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{t_{kn}(x-l)}{\rho}, z \right\| \right) \right]^{p_k} + \frac{K}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{1}{\rho}, z \right\| \right) \right]^{p_k}.$$

Since $\sup_k [M_k(t)]p_k < \infty$, we can take that $\sup_k [M_k(t)]p_k = T$. Hence we get $x \in w_\sigma^\infty [\mathcal{M}, p, ||.,||]_\theta$. □

Theorem 4 Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function which satisfies $\Delta_2$-condition for all $k$, then 
$$w_\sigma [p, ||.,||]_\theta \subset w_\sigma [\mathcal{M}, p, ||.,||]_\theta.$$

Proof. Let $x \in w_\sigma [p, ||.,||]_\theta$. Then we have 
$$T_r = \frac{1}{h_r} \sum_{k \in I_r} ||t_{kn}(x-l), z||^{p_k} \to \infty \text{ as } r \to \infty \text{ uniformly in } n, \text{ for some } l.$$

Let $\epsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$ for $0 \leq t \leq \delta$ for all $k$. So that
$$\frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{t_{kn}(x-l)}{\rho}, z \right\| \right) \right]^{p_k} = \frac{1}{h_r} \sum_{k \in I_r, |t_{kn}(x-l)| \leq \delta} \left[ M_k \left( \left\| \frac{t_{kn}(x-l)}{\rho}, z \right\| \right) \right]^{p_k} + \frac{1}{h_r} \sum_{k \in I_r, |t_{kn}(x-l)| \leq \delta} \left[ M_k \left( \left\| \frac{t_{kn}(x-l)}{\rho}, z \right\| \right) \right]^{p_k}.$$

For the first summation in the right hand side of the above equation, we have
$$\sum_{k \in I_r} \leq \epsilon^H$$
by using continuity of $M_k$ for all $k$. For the second summation, we write
$$||t_{kn}(x-l), z|| \leq 1 + \frac{t_{kn}(x-l)}{\delta}, z||.$$

For the first summation in the right hand side of the above equation, we have
Since $M_k$ is non-decreasing and convex for all $k$, it follows that
\[
M_k\left(\|t_{kn}(x-1), z\|\right) < M_k\left(1 + \left\|\frac{t_{kn}(x-1)}{\delta}, z\right\|\right)
\]
\[
\leq \frac{1}{2} M_k(2) + \frac{1}{2} M_k\left(\left(2\right) \left\|\frac{t_{kn}(x-1)}{\delta}, z\right\|\right).
\]

Since $M_k$ satisfies $\Delta_2$-condition for all $k$, we can write
\[
M_k\left(\|t_{kn}(x-1), z\|\right) \leq \frac{1}{2} L \left\|\frac{t_{kn}(x-1)}{\delta}, z\right\| M_k(2) + \frac{1}{2} L \left\|\frac{t_{kn}(x-1)}{\delta}, z\right\| M_k(2).
\]

So we write
\[
\frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left(\left\|\frac{t_{kn}(x-1)}{\rho}, z\right\|\right)\right]^{p_k} \leq e^{H} + \left[\max(1, LM_k(2))\right] \delta H r.
\]

Letting $r \to \infty$, it follows that $x \in w_\sigma^\infty[M, p, \|\|, \|\]_\theta$.
This completes the proof. \hfill \Box

**Theorem 5** Let $M = (M_k)$ be Musielak-Orlicz function. Then the following statements are equivalent:

(i) $w_\sigma^\infty[p, \|\|, \|\]_\theta \subset w_\sigma^0[M, p, \|\|, \|\]_\theta$,
(ii) $w_\sigma^0[p, \|\|, \|\]_\theta \subset w_\sigma^\infty[M, p, \|\|, \|\]_\theta$,
(iii) $\sup_r \frac{1}{h_r} \sum_{k \in I_r} [M_k(t)]^{p_k} < \infty$ for all $t > 0$.

**Proof.** (i) $\Rightarrow$ (ii) We have only to show that $w_\sigma^0[p, \|\|, \|\]_\theta \subset w_\sigma^\infty[p, \|\|, \|\]_\theta$. Let $x \in w_\sigma^0[p, \|\|, \|\]_\theta$. Then there exists $r > r_\epsilon$, for $\epsilon > 0$, such that
\[
\frac{1}{h_r} \sum_{k \in I_r} \left\|\frac{t_{kn}(x)}{\rho}, z\right\|^{p_k} < \epsilon.
\]

Hence there exists $H > 0$ such that
\[
\sup_{r, n} \frac{1}{h_r} \sum_{k \in I_r} \left\|\frac{t_{kn}(x)}{\rho}, z\right\|^{p_k} < H.
\]
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for all \( n \) and \( r \). So we get \( x \in w^\infty_\sigma[p,||.||]_\theta \).

(ii) \( \implies \) (iii) Suppose that (iii) does not hold. Then for some \( t > 0 \)

\[
\sup_r \frac{1}{h_r} \sum_{k \in I_r} [M_k(t)]^{p_k} = \infty
\]

and therefore we can find a subinterval \( I_{r(m)} \) of the set of interval \( I_r \) such that

\[
\frac{1}{h_{r(m)}} \sum_{k \in I_{r(m)}} \left( M_k \left( \frac{1}{m} \right) \right)^{p_k} > m, \ m = 1, 2,
\]  

(2)

Let us define \( x = (x_k) \) as follows, \( x_k = \frac{1}{m} \) if \( k \in I_{r(m)} \) and \( x_k = 0 \) if \( k \not\in I_{r(m)} \). Then \( x \in w^\infty_\sigma[p,||.||]_\theta \) but by eqn. (2), \( x \not\in w^\infty_\sigma[M, p, ||.||]_\theta \) which contradicts (ii). Hence (iii) must hold. (iii) \( \implies \) (i) Suppose (i) not holds, then for \( x \in w^\infty_\sigma[p,||.||]_\theta \), we have

\[
\sup_r \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \frac{1}{m} \right) \right]^{p_k} = \infty
\]

which contradicts (iii). Hence (i) must hold. \( \Box \)

**Theorem 6** Let \( M = (M_k) \) be Musielak-Orlicz function. Then the following statements are equivalent:

(i) \( w^\infty_\sigma[M, p, ||.||]_\theta \subset w^\infty_\sigma[p,||.||]_\theta \),

(ii) \( w^\infty_\sigma[M, p, ||.||]_\sigma \subset w^\infty_\sigma[p,||.||]_\sigma \),

(iii) \( \inf_{t > 0} \sum_{k \in I_r} [M_k(t)]^{p_k} > 0 \) for all \( t > 0 \).

**Proof.** (i) \( \implies \) (ii) : It is easy to prove.

(ii) \( \implies \) (iii) Suppose that (iii) does not hold. Then

\[
\inf_{t > 0} \frac{1}{h_r} \sum_{k \in I_r} [M_k(t)]^{p_k} = 0
\]

for some \( t > 0 \),
and we can find a subinterval $I_{r(m)}$ of the set of interval $I_r$ such that

$$\frac{1}{h_r} \sum_{k \in I_{r(m)}} [M_k(m)]^{p_k} < \frac{1}{m}, \quad m = 1, 2, \ldots$$

(4)

Let us define $x_k = m$ if $k \in I_{r(m)}$ and $x_k = 0$ if $k \not\in I_{r(m)}$. Thus by eqn.(4),

$$x \in w_\sigma^\infty[\mathcal{M}, p, ||., ||] \quad \text{but} \quad x \not\in w_\sigma^\infty[p, ||., ||]$$

which contradicts (ii). Hence (iii) must hold.

(iii) $\implies$ (i) It is obvious. □

**Theorem 7** Let $\mathcal{M} = \{M_k\}$ be Musielak-Orlicz function. Then $w_\sigma^\infty[\mathcal{M}, p, ||., ||] \subset w_\sigma^\infty[p, ||., ||]$ if and only if

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} [M_k(t)]^{p_k} = \infty$$

(5)

**Proof.** Let $w_\sigma^\infty[\mathcal{M}, p, ||., ||] \subset w_\sigma^\infty[p, ||., ||]$. Suppose that eqn. (5) does not hold. Therefore there is a subinterval $I_{r(m)}$ of the set of interval $I_r$ and a number $t_o > 0$, where $t_o = ||t_kn(x), z||$ for all $k$ and $n$, such that

$$\frac{1}{h_{r(m)}} \sum_{k \in I_{r(m)}} [M_k(t_o)]^{p_k} \leq M < \infty, \quad m = 1, 2, \ldots$$

(6)

Let us define $x_k = t_o$ if $k \in I_{r(m)}$ and $x_k = 0$ if $k \not\in I_{r(m)}$. Then, by eqn. (6), $x \in w_\sigma^\infty[\mathcal{M}_k, p, ||., ||]$. But $x \not\in w_\sigma^\infty[p, ||., ||]$. Hence eqn. (5) must hold.

Conversely, suppose that eqn. (5) hold and that $x \in w_\sigma^\infty[\mathcal{M}_k, p, ||., ||]$. Then for each $r$ and $n$

$$\frac{1}{h_r} \sum_{k \in I_r} [M_k(t_kn(x))]^{p_k} \leq M < \infty.$$

(7)

Now suppose that $x \not\in w_\sigma^\infty[p, ||., ||]$. Then for some number $\epsilon > 0$ and for a subinterval $I_{r+1}$ of the set of interval $I_r$, there is $k_o$ such that $||t_kn(x), z||^{p_k} > \epsilon$ for $k \geq k_o$. From the properties of sequence of Orlicz functions, we obtain

$$[M_k\left(\frac{\epsilon}{p}\right)]^{p_k} \leq [M_k\left(||t_kn(x), z||\right)]^{p_k}$$

which contradicts eqn.(6), by using eqn. (7). This completes the proof. □
Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz function

References


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