Abstract. Making use of convolution product, we introduce a novel subclass of \(p\)-valent analytic functions with negative coefficients and obtain coefficient bounds, extreme points and radius of starlikeness for functions belonging to the generalized class \(TP^{k,p}_{b,\mu}(\lambda, \alpha, \beta)\). We also derive results for the modified Hadamard products of functions belonging to the class \(TP^{k,p}_{b,\mu}(\lambda, \alpha, \beta)\).

1 Introduction

Denote by \(A_p\) the class of functions \(f\) normalized by

\[
f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}z^{p+k}, \quad (p \in \mathbb{N} = 1, 2, 3, ...)
\]

which are analytic and \(p\)-valent in the open disc \(U = \{z : z \in \mathbb{C}, |z| < 1\}\). Denote by \(T_p\) a subclass of \(A_p\) consisting of functions of the form

\[
f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k}z^{p+k}, \quad (a_{p+k} \geq 0; p \in \mathbb{N} = 1, 2, 3, ..., z \in U).
\]
For functions $f \in A_p$ given by (1) and $g \in A_p$ given by $g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k}z^{p+k}$, we define the Hadamard product (or convolution) of $f$ and $g$ by

$$(f \ast g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}b_{p+k}z^{p+k} = (g \ast f)(z), \quad z \in U. \quad (3)$$

The following we recall a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (see [23])

$$\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k + a)^s} \quad (4)$$

where, as usual, $Z_0^{-} := \mathbb{Z}\{0\} \setminus \mathbb{N}$ ($\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, ..., \}; \mathbb{N} := \{1, 2, 3, ..., \}$). Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ are found in the recent investigations by Choi and Srivastava [4], Ferreira and Lopez [5], Garg et al. [7], Lin and Srivastava [10], Lin et al. [11], and others.

For the class of analytic functions denote by $A$ consisting of functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ Srivastava and Attiya [22] (see also Raducanu and Srivastava [17], and Prajapat and Goyal [14]) introduced and investigated the linear operator:

$$J_{\mu, b} : A_p \rightarrow A_p$$

defined in terms of the Hadamard product (or convolution) by

$$J_{\mu, b} f(z) = G_{b, \mu} \ast f(z) \quad (5)$$

(z \in U; $b \in \mathbb{C \setminus \{Z_0^{-}\}}; \mu \in \mathbb{C}; f \in A$), where, for convenience,

$$G_{\mu, b}(z) := (1 + b)^{\mu} \Phi(z, \mu, b) - b^{-\mu} \quad (z \in U). \quad (6)$$

It is easy to observe from (given earlier by [14], [17]) (1), (5) and (6) that

$$J_{b, \mu}^{1+b} f(z) = z + \sum_{k=2}^{\infty} \left(1 + b\right)^{\mu} (k + b)^{\mu} a_k z^k. \quad (7)$$

Motivated essentially by the above-mentioned Srivastava-Attiya operator, we define the operator

$$J_{b, \mu}^{n_p} : A_p \rightarrow A_p$$
which is defined as

\[ J_{b,\mu}^{k,p}f(z) = z^p + \sum_{k=1}^{\infty} C_{b}^{\mu}(k, p) a_{p+k} z^{p+k} \quad (z \in \mathbb{U}; f(z) \in \mathcal{A}_p) \]  

(8)

where

\[ C_{b}^{\mu}(k, p) = \left| \left( \frac{p + b}{k + p + b} \right)^{\mu} \right| \]  

(9)

and (throughout this paper unless otherwise mentioned) the parameters \( \mu, b \) are constrained as

\[ b \in \mathbb{C} \setminus \{ \mathbb{Z}^+ \}; \mu \in \mathbb{C} \quad \text{and} \quad p, \in \mathbb{N}. \]

1. For \( \mu = 1 \) and \( b = \nu (\nu > -1) \) generalized Libera Bernardi integral operators [16]

\[ J_{\nu,1}^{k,p}f(z) := \frac{p + \nu}{z^\nu} \int_0^z t^{\nu-1} f(t) \, dt := z + \sum_{k=1}^{\infty} \left( \frac{\nu + p}{k + p + \nu} \right) a_{p+k} z^{p+k} := \mathcal{L}_p f(z). \]  

(10)

2. For \( \mu = \sigma (\sigma > 0) \) and \( b = 1 \) Jung-Kim-Srivastava integral operator [12]

\[ J_{1,\sigma}^{k,p}f(z) := z + \sum_{k=1}^{\infty} \left( \frac{1 + p}{k + p + 1} \right)^{\sigma} a_{p+k} z^{p+k} := \mathcal{I}_\sigma f(z) \]  

(11)

closely related to some multiplier transformation studied by Flett[6]. Making use of the operator \( J_{b,\mu}^{k,p} \), and motivated by earlier works of [1, 2, 3, 8, 9, 15, 13, 20, 21, 24, 25, 26], we introduced a new subclass of analytic functions with negative coefficients and discuss some some usual properties of the geometric function theory of this generalized function class.

For \( 0 \leq \lambda \leq 1, 0 \leq \alpha < 1 \) and \( \beta \geq 0 \), we let \( P_{b,\mu}^{k,p}(\lambda, \alpha, \beta) \) be the subclass of \( \mathcal{A}_p \) consisting of functions of the form (1) and satisfying the inequality

\[
\begin{aligned}
\text{Re} \left\{ \frac{(1 - \lambda + \frac{\lambda}{p}) z (J_{b,\mu}^{k,p}f(z))'}{p(1 - \lambda)} + \frac{\lambda z^2 (J_{b,\mu}^{k,p}f(z))''}{p(1 - \lambda)} - \alpha \right\} > \beta \\
\end{aligned}
\]  

(12)
where \( z \in U \), \( \mathcal{S}^{k,p}_{b,\mu} f(z) \) is given by (8). We further let \( TP^{k,p}_{b,\mu}(\lambda, \alpha, \beta) = P^{k,p}_{b,\mu}(\lambda, \alpha, \beta) \cap T_p \).

In particular, for \( 0 \leq \lambda \leq 1 \), the class \( TP^{k,p}_{b,\mu}(\lambda, \alpha, \beta) \) provides a transition from \( k \)-uniformly starlike functions to \( k \)-uniformly convex functions.

**Example 1** If \( \lambda = 0 \), then

\[
TP^{k,p}_{b,\mu}(0, \alpha, \beta) \equiv TS^{k,p}_{b,\mu}(\alpha, \beta) := \text{Re} \left\{ \frac{1}{p} \frac{z(J^{k,p}_{b,\mu} f(z))'}{J^{k,p}_{b,\mu} f(z)} - \alpha \right\} > \beta \left| \frac{1}{p} \frac{z(J^{k,p}_{b,\mu} f(z))'}{J^{k,p}_{b,\mu} f(z)} - 1 \right|, z \in U.
\]

**Example 2** If \( \lambda = 1 \), then

\[
TP^{k,p}_{b,\mu}(1, \alpha, \beta) \equiv UTC^{k,p}_{b,\mu}(\alpha, \beta) := \text{Re} \left\{ \frac{1}{p} \left[ 1 + \frac{z(J^{k,p}_{b,\mu} f(z))''}{(J^{k,p}_{b,\mu} f(z))'} \right] - \alpha \right\} > \beta \left| \frac{1}{p} \left[ 1 + \frac{z(J^{k,p}_{b,\mu} f(z))''}{(J^{k,p}_{b,\mu} f(z))'} \right] - 1 \right|, z \in U.
\]

**Example 3** For \( \mu = 1, b = q (q > -1) \) and \( f(z) \) is as defined in (10) is in \( L^p_{\nu}(\lambda, \alpha, \beta) \) if

\[
\text{Re} \left( \left( 1 - \lambda + \frac{\lambda}{p} \right) z(L^p_{\nu} f(z))' + \frac{\lambda}{p} z^2(L^p_{\nu} f(z))'' \right) > \beta \left| \left( 1 - \lambda + \frac{\lambda}{p} \right) z(L^p_{\nu} f(z))' + \frac{\lambda}{p} z^2(L^p_{\nu} f(z))'' \right|, z \in U.
\]

Also, let \( L^p_{\nu}(\lambda, \alpha, \beta) \cap T_p = T L^p_{\nu}(\lambda, \alpha, \beta) \).

**Example 4** For \( \mu = \sigma (\sigma > 0) \), \( b = 1 \) and \( f(z) \) is defined in (11), is in \( \mathcal{I}^p_{\sigma}(\lambda, \alpha, \beta) \) if

\[
\text{Re} \left( \left( 1 - \lambda + \frac{\lambda}{p} \right) z(I^p_{\sigma} f(z))' + \frac{\lambda}{p} z^2(I^p_{\sigma} f(z))'' \right) > \beta \left| \left( 1 - \lambda + \frac{\lambda}{p} \right) z(I^p_{\sigma} f(z))' + \frac{\lambda}{p} z^2(I^p_{\sigma} f(z))'' \right|, z \in U.
\]

Also, let \( I^p_{\sigma}(\lambda, \alpha, \beta) \cap T_p = T I^p_{\sigma}(\lambda, \alpha, \beta) \).
The main object of this paper is to study the coefficient bounds, extreme points and radius of starlikeness for functions belonging to the generalized class $TP^{k,p}_{b,\mu}(\lambda, \alpha, \beta)$ employing the technique of Silverman[18] and also derive results for the modified Hadamard products of functions belonging to the class $TP^{k,p}_{b,\mu}(\lambda, \alpha, \beta)$ using the techniques of Schild and Silverman [19].

## 2 Coefficient Bounds

In this section we obtain a necessary and sufficient condition for functions $f(z)$ in the classes $P^{k,p}_{b,\mu}(\lambda, \alpha, \beta)$ and $TP^{k,p}_{b,\mu}(\lambda, \alpha, \beta)$.

**Theorem 1** A function $f(z)$ of the form (1) is in $P^{k,p}_{b,\mu}(\lambda, \alpha, \beta)$ if

\[
\sum_{k=1}^{\infty} |p + k\lambda| |k(1 + \beta) + p(1 - \alpha)| |C^k_b(k, p)||a_{p+k}| \leq p^2(1 - \alpha), \tag{1}
\]

$0 \leq \lambda \leq 1$, $-1 \leq \alpha < 1$, $\beta \geq 0$.

**Proof.** It suffices to show that

\[
\beta \left| \frac{(1 - \lambda + \frac{1}{p})z(J^{k,p}_{b,\mu}f(z))' + \frac{\lambda}{p}z^2(J^{k,p}_{b,\mu}f(z))''}{p(1 - \lambda)J^{k,p}_{b,\mu}f(z) + \lambda z(J^{k,p}_{b,\mu}f(z))'} - 1 \right| - \text{Re} \left\{ \frac{(1 - \lambda + \frac{1}{p})z(J^{k,p}_{b,\mu}f(z))' + \frac{\lambda}{p}z^2(J^{k,p}_{b,\mu}f(z))''}{p(1 - \lambda)J^{k,p}_{b,\mu}f(z) + \lambda z(J^{k,p}_{b,\mu}f(z))'} - 1 \right\} \leq 1 - \alpha
\]

We have

\[
\beta \left| \frac{(1 - \lambda + \frac{1}{p})z(J^{k,p}_{b,\mu}f(z))' + \frac{\lambda}{p}z^2(J^{k,p}_{b,\mu}f(z))''}{p(1 - \lambda)J^{k,p}_{b,\mu}f(z) + \lambda z(J^{k,p}_{b,\mu}f(z))'} - 1 \right| - \text{Re} \left\{ \frac{(1 - \lambda + \frac{1}{p})z(J^{k,p}_{b,\mu}f(z))' + \frac{\lambda}{p}z^2(J^{k,p}_{b,\mu}f(z))''}{p(1 - \lambda)J^{k,p}_{b,\mu}f(z) + \lambda z(J^{k,p}_{b,\mu}f(z))'} - 1 \right\} \leq (1 + \beta) \left| \frac{(1 - \lambda + \frac{1}{p})z(J^{k,p}_{b,\mu}f(z))' + \frac{\lambda}{p}z^2(J^{k,p}_{b,\mu}f(z))''}{p(1 - \lambda)J^{k,p}_{b,\mu}f(z) + \lambda z(J^{k,p}_{b,\mu}f(z))'} - 1 \right|
\]

\[
0 \leq \sum_{k=1}^{\infty} k[p + k\lambda]|C^k_b(k, p)||a_{p+k}| \leq p - \sum_{k=1}^{\infty} |p + k\lambda||C^k_b(k, p)||a_{p+k}|.
\]
This last expression is bounded above by \((1 - \alpha)\) if
\[
\sum_{k=1}^{\infty} |p + k\lambda|[k(1 + \beta) + p(1 - \alpha)] |C_0^\mu(k, p)| a_{p+k} \leq p^2(1 - \alpha)
\]
and hence the proof is complete. \(\square\)

**Theorem 2** A necessary and sufficient condition for \(f(z)\) of the form (2) to be in the class \(TP_{k,p}^\mu(\lambda, \alpha, \beta)\), \(-1 \leq \alpha < 1, 0 \leq \lambda \leq 1, \beta \geq 0\) is that
\[
\sum_{k=1}^{\infty} |p + k\lambda|[k(1 + \beta) + p(1 - \alpha)] |C_0^\mu(k, p)| a_{p+k} \leq p^2(1 - \alpha),
\]

**Proof.** In view of Theorem 1, we need only to prove the necessity. If \(f \in P_{k,p}^\mu(\lambda, \alpha, \beta)\) and \(z\) is real then
\[
1 - \sum_{k=1}^{\infty} \frac{[p+k\lambda]}{p} |C_0^\mu(k, p)| a_{p+k} |z|^k - \alpha \geq \beta \sum_{k=1}^{\infty} \frac{k[p+k\lambda]}{p} |C_0^\mu(k, p)| a_{p+k} |z|^k
\]
Letting \(z \to 1\) along the real axis, we obtain the desired inequality
\[
\sum_{k=1}^{\infty} |p + k\lambda|[k(1 + \beta) + p(1 - \alpha)] |C_0^\mu(k, p)| a_{p+k} \leq p^2(1 - \alpha).
\]

\(\square\)

In view of the Examples 1 to 4 in Section 1 and Theorem 2, we have following corollaries for the classes defined in these examples.

**Corollary 1** A necessary and sufficient condition for \(f(z)\) of the form (2) to be in the class \(TS_{k,p}^\mu(\alpha, \beta)\), \(0 \leq \alpha < 1, \beta \geq 0\) is that
\[
\sum_{k=1}^{\infty} [k(1 + \beta) + p(1 - \alpha)] |C_0^\mu(k, p)| a_{p+k} \leq p(1 - \alpha),
\]

**Corollary 2** A necessary and sufficient condition for \(f(z)\) of the form (2) to be in the class \(UCT_{k,p}^\mu(\alpha, \beta)\), \(0 \leq \alpha < 1, \beta \geq 0\) is that
\[
\sum_{k=1}^{\infty} (p + k)[k(1 + \beta) + p(1 - \alpha)] |C_0^\mu(k, p)| a_{p+k} \leq p^2(1 - \alpha),
\]
Corollary 3 A necessary and sufficient condition for \( f(z) \) of the form (2) to be in the class \( TL_{\nu}^k(\lambda, \alpha, \beta) \), \( 0 \leq \alpha < 1, \beta \geq 0 \) is that

\[
\sum_{k=1}^{\infty} (p+k\lambda)[k(1+\beta)+p(1-\alpha)] \left( \frac{p+\nu}{k+p+\nu} \right) a_{p+k} \leq p^2(1-\alpha).
\]

Corollary 4 A necessary and sufficient condition for \( f(z) \) of the form (2) to be in the class \( TT_p(\lambda, \alpha, \beta) \), \( 0 \leq \alpha < 1, \beta \geq 0 \) is that

\[
\sum_{k=1}^{\infty} (p+k\lambda)[k(1+\beta)+p(1-\alpha)] \left( \frac{1+p}{k+p+1} \right)^\sigma a_{p+k} \leq p^2(1-\alpha).
\]

Corollary 5 If \( f \in TP_{b,\mu}^k(\lambda, \alpha, \beta) \), then

\[
a_{p+k} \leq \frac{p^2(1-\alpha)}{[p+k\lambda][k(1+\beta)+p(1-\alpha)]C_b^\mu(k,p)}, \quad k \geq 1,
\]

where \( 0 \leq \lambda \leq 1, -1 \leq \alpha < 1 \) and \( \beta \geq 0 \). Equality in (3) holds for the function

\[
f(z) = z - \frac{p^2(1-\alpha)}{[p+k\lambda][k(1+\beta)+p(1-\alpha)]C_b^\mu(k,p)} z^{p+k} \quad (p \in \mathbb{N}).
\]

It is of interest to note that, when \( p = 1 \) and \( k = n-1 \), the above results reduces to the results studied in [2, 8, 9, 20, 21] Similarly many known results can be obtained as particular cases of the following theorems, so we omit stating the particular cases for the following theorems.

3 Closure Properties

Theorem 1 Let

\[
f_p(z) = z^p \quad (p \in \mathbb{N}) \quad \text{and} \quad f_{p+k}(z) = z^p - \frac{p^2(1-\alpha)}{[p+k\lambda][k(1+\beta)+p(1-\alpha)]C_b^\mu(k,p)} z^{p+k}.
\]

Then \( f \in TP_{b,\mu}^k(\lambda, \alpha, \beta) \), if and only if it can be expressed in the form

\[
f(z) = \sum_{k=0}^{\infty} \omega_{p+k} f_{p+k}(z), \quad \omega_{p+k} \geq 0, \quad \sum_{k=0}^{\infty} \omega_{p+k} = 1.
\]
Proof. Let us suppose that \( f(z) \) is given by (2), that is by
\[
f(z) = z^p - \sum_{k=1}^{\infty} \frac{p^2(1-\alpha)}{|p+k\lambda|[k(1+\beta)+p(1-\alpha)]|C_B^p(k,p)|^\omega_{p+k}} z^{p+k}.
\]
Then, since
\[
\sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_B^p(k,p)|}{p^2(1-\alpha)[p+k\lambda][k(1+\beta)+p(1-\alpha)]} \omega_{p+k} = \sum_{k=1}^{\infty} \omega_{p+k} = 1 - \omega_p \leq 1.
\]
Thus \( f \in TP_{b,\mu}^{k,p}(\lambda, \alpha, \beta) \). Conversely, let us have \( f \in TP_{b,\mu}^{k,p}(\lambda, \alpha, \beta) \). Then by using (3), we set
\[
\omega_{p+k} = \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_B^p(k,p)|}{p^2(1-\alpha)} a_{p+k}, \quad (k \in \mathbb{N})
\]
and \( \omega_p = 1 - \sum_{k=1}^{\infty} \omega_{p+k} \), we can readily see that \( f(z) \) can be expressed precisely as in (1). This evidently completes the proof of Theorem 1.

\[\square\]

Theorem 2 The class \( TP_{b,\mu}^{k,p}(\lambda, \alpha, \beta) \) is a convex set.

Proof. Let the function
\[
f_j(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,j} z^{p+k}, \quad (a_{p+k,j} \geq 0, p \in \mathbb{N}; \ j = 1, 2,...) \quad (3)
\]
be in the class \( TP_{b,\mu}^{k,p}(\lambda, \alpha, \beta) \). It sufficient to show that the function \( h(z) \) defined by
\[
h(z) = \eta f_1(z) + (1-\eta)f_2(z), \quad 0 \leq \eta \leq 1,
\]
is in the class \( TP_{b,\mu}^{k,p}(\lambda, \alpha, \beta) \). Since
\[
h(z) = z^p - \sum_{k=1}^{\infty} [\eta a_{p+k,1} + (1-\eta) a_{p+k,2}] z^{p+k},
\]
an easy computation with the aid of Theorem 2 gives,

\[ \sum_{k=1}^{\infty} [p + k\lambda][k(1 + \beta) + p(1 - \alpha)]\eta|C^\mu_0(k, p)|a_{p+k,1} \]
\[ + \sum_{k=1}^{\infty} [p + k\lambda][k(1 + \beta) + p(1 - \alpha)][1 - \eta]|C^\mu_0(k, p)|a_{p+k,2} \]
\[ \leq p^2\eta(1 - \alpha) + p^2(1 - \eta)(1 - \alpha) \]
\[ \leq p^2(1 - \alpha), \]

which implies that \( h \in TP_{b,\mu}^k(\lambda, \alpha, \beta) \). Hence \( TP_{b,\mu}^k(\lambda, \alpha, \beta) \) is convex. \( \square \)

Now we provide the radii of \( p \)-valently close-to-convexity, starlikeness and convexity for the class \( TP_{b,\mu}^k(\lambda, \alpha, \beta) \).

**Theorem 3** Let the function \( f(z) \) defined by (2) be in the class \( TP_{b,\mu}^k(\lambda, \alpha, \beta) \). Then \( f(z) \) is \( p \)-valently close-to-convex of order \( \delta \) \( (0 \leq \delta < p) \) in the disc \( |z| < r_1 \), where

\[ r_1 := \inf_{k \in \mathbb{N}} \left[ \left( 1 - \delta \right)[k(1 + \beta) + p(1 - \alpha)][p + k\lambda]|C^\mu_0(k, p)| \right]^{\frac{1}{k}}. \]

The result is sharp, with extremal function \( f(z) \) given by (1).

**Proof.** Given \( f \in T_p \), and \( f \) is close-to-convex of order \( \delta \), we have

\[ \left| \frac{f'(z)}{z^{p-1}} - p \right| < p - \delta. \]

(5)

For the left hand side of (5) we have

\[ \left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{k=1}^{\infty} (p + k)a_{p+k}|z|^k. \]

The last expression is less than \( p - \delta \) if

\[ \sum_{k=1}^{\infty} \frac{p + k}{p - \delta}a_{p+k}|z|^k < 1. \]

Using the fact, that \( f \in TP_{b,\mu}^k(\lambda, \alpha, \beta) \) if and only if

\[ \sum_{k=1}^{\infty} [p + k\lambda][k(1 + \beta) + p(1 - \alpha)]|C^\mu_0(k, p)|a_n \leq 1, \]
We can say (5) is true if

$$\frac{p + k}{p - \delta} |z|^k \leq \frac{[p + k\lambda][k(1 + \beta) + p(1 - \alpha)]|C_b^\mu(k, p)|}{p^2(1 - \alpha)} a_n$$

Or, equivalently,

$$|z|^k = \left[ \frac{[p - \delta][p + k\lambda][k(1 + \beta) + p(1 - \alpha)]|C_b^\mu(k, p)|}{p^2(p + k)(1 - \alpha)} \right].$$

the last inequality leads us immediately to the disc $|z| < r_1$, where $r_1$ given by (4) and the proof of Theorem 3 is completed. $\square$

**Theorem 4** If $f \in TP_{b, \mu}^k(\lambda, \alpha, \beta)$, then

(i) $f$ is $p$-valently starlike of order $\delta (0 \leq \delta < p)$ in the disc $|z| < r_2$; that is,

$$Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \delta,$$

where

$$r_2 = \inf_{k \in \mathbb{N}} \left[ \left( \frac{p - \delta}{p + k - \delta} \right) \frac{[p + k\lambda][k(1 + \beta) + p(1 - \alpha)]|C_b^\mu(k, p)|}{p^2(1 - \alpha)} \right]^{\frac{1}{k}}.$$ 

(ii) $f$ is convex of order $\delta (0 \leq \delta < p)$ in the unit disc $|z| < r_3$, that is

$$Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \delta,$$

where

$$r_3 = \inf_{k \in \mathbb{N}} \left[ \left( \frac{p - \delta}{(k + p)(p + k - \delta)} \right) \frac{[p + k\lambda][k(1 + \beta) + p(1 - \alpha)]|C_b^\mu(k, p)|}{p^2(1 - \alpha)} \right]^{\frac{1}{k}}.$$ 

Each of these results are sharp for the extremal function $f(z)$ given by (1).

**Proof.** (i) Given $f \in T_p$, and $f$ is starlike of order $\delta$, we have

$$\left| \frac{zf'(z)}{f(z)} - p \right| < p - \delta.$$ 

For the left hand side of (8) we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \sum_{k=1}^{\infty} k a_{p+k} |z|^k \leq \frac{\sum_{k=1}^{\infty} k a_{p+k} |z|^k}{1 - \sum_{k=1}^{\infty} a_{p+k} |z|^k}.$$
The last expression is less than $p - \delta$ if

$$\sum_{k=1}^{\infty} \frac{k + p - \delta}{p - \delta} a_{p+k} |z|^k < 1.$$ 

Using the fact, that $f \in TP_{b,\nu}^{k,p}(\lambda, \alpha, \beta)$ if and only if

$$\sum_{k=1}^{\infty} \frac{[p + k\lambda][k(1 + \beta) + p(1 - \alpha)]}{p^2(1 - \alpha)} a_{p+k}|C_{\mu}^b(k,p)| \leq 1.$$ 

We can say (8) is true if

$$\frac{p + k - \delta}{p - \delta} |z|^k < \frac{[p + k\lambda][k(1 + \beta) + p(1 - \alpha)]|C_{\mu}^b(k,p)|}{p^2(1 - \alpha)}.$$ 

Or, equivalently,

$$|z|^k = \left[ \frac{p - \delta}{p + k - \delta} \frac{[p + k\lambda][k(1 + \beta) + p(1 - \alpha)]|C_{\mu}^b(k,p)|}{p^2(1 - \alpha)} \right]$$

which yields the starlikeness of the family.

(ii) Using the fact that $f$ is convex if and only if $zf'$ is starlike, we can prove (ii), on lines similar to the proof of (i).

\[\square\]

4 Convolution Results

Let the functions

$$f_j(z) = z^p + \sum_{k=1}^{\infty} a_{j,p+k} z^{p+k}, \quad (p \in \mathbb{N} = 1, 2, 3, \ldots) (j = 1, 2) \quad (9)$$

then the modified Hadamard product of $f_1(z)$ and $f_2(z)$ is given by

$$(f_1 * f_2)(z) = z^p - \sum_{n=2}^{\infty} a_{1,p+k} a_{2,p+k} z^{p+k} = (f_2 * f_1)(z), (a_{1,p+k} \geq 0; a_{2,p+k} \geq 0).$$

Using the techniques of we prove the following results.
Theorem 5 For functions $f_j(z)(j = 1, 2)$ defined by (9), be in the class $TP_{b,µ}^{k,p}(\lambda, \alpha, \beta)$. Then $(f_1 * f_2) \in TP_{b,µ}^{k,p}(\lambda, \xi, \beta)$ where

$$\xi = 1 - \frac{p^2(1 - \alpha)^2(1 + \beta)}{|p + \lambda|[1 + \beta] + p(1 - \alpha)|C_b^1(1, p)| - p^3(1 - \alpha)^2}$$  \hspace{1cm} (10)

where $C_b^1(1, p)$ is given by (9).

Proof. Employing the technique used earlier by Schild and Silverman[19], we need to find the largest $\xi$ such that

$$\sum_{k=1}^{\infty} \frac{|p + k\lambda|[k(1 + \beta) + p(1 - \xi)]|C_b^1(k, p)|}{|p^2(1 - \alpha)|} a_{1,p+k} a_{2,p+k} \leq 1, \quad (0 \leq \xi < 1)$$

for $f_j \in TP_{b,µ}^{k,p}(\lambda, \alpha, \beta)(j = 1, 2)$ where $\xi$ is defined by (10). On the other hand, under the hypothesis, it follows from (1) and the Cauchy’s-Schwarz inequality that

$$\sum_{k=1}^{\infty} \frac{|p + k\lambda|[k(1 + \beta) + p(1 - \alpha)]|C_b^1(k, p)|}{|p^2(1 - \alpha)|} \sqrt{a_{1,p+k} a_{2,p+k}} \leq 1.$$  \hspace{1cm} (11)

Thus we need to find the largest $\xi$ such that

$$\sum_{k=1}^{\infty} \frac{|p + k\lambda|[k(1 + \beta) + p(1 - \alpha)]|C_b^1(k, p)|}{|p^2(1 - \alpha)|} a_{1,p+k} a_{2,p+k} \leq \sum_{k=1}^{\infty} \frac{|p + k\lambda|[k(1 + \beta) + p(1 - \alpha)]|C_b^1(k, p)|}{|p^2(1 - \alpha)|} \sqrt{a_{1,p+k} a_{2,p+k}}$$

or, equivalently that

$$\sqrt{a_{1,p+k} a_{2,p+k}} \leq \frac{(1 - \xi)[k(1 + \beta) + p(1 - \alpha)]}{(1 - \alpha)[k(1 + \beta) + p(1 - \xi)]}, \quad (k \geq 1).$$

Hence by making use of the inequality (11), it is sufficient to prove that

$$\frac{p^2(1 - \alpha)}{|p + k\lambda|[k(1 + \beta) + p(1 - \alpha)]|C_b^1(k, p)|} \leq \frac{(1 - \xi)[k(1 + \beta) + p(1 - \alpha)]}{(1 - \alpha)[k(1 + \beta) + p(1 - \xi)]}$$

which yields

$$\xi = \Psi(k) = 1 - \frac{kp^2(1 - \alpha)^2(1 + \beta)}{|p + k\lambda|[k(1 + \beta) + p(1 - \alpha)]|C_b^1(k, p)| - p^3(1 - \alpha)^2}$$  \hspace{1cm} (12)
for \( k \geq 1 \) is an increasing function of \( k \) and letting \( k = 1 \) in (12), we have

\[
\delta = \Psi(1) = 1 - \frac{p^2(1 - \alpha)^2(1 + \beta)}{[p + \lambda][(1 + \beta) + p(1 - \alpha)]^2|C_{b}^{k}(1, p)| - p^3(1 - \alpha)^2}
\]

where \( C_{b}^{k}(1, p) \) is given by (9).

\[ \square \]

**Theorem 6** Let the function \( f(z) \) defined by (2) be in the class \( TP_{k,p}^{k^p}(\lambda, \alpha, \beta) \).

Also let \( g(z) = z^p - \sum_{k=1}^{\infty} b_{p+k}z^{p+k} \) for \( |b_{p+k}| \leq 1 \). Then \( (f \ast g) \in TP_{k,p}^{k^p}(\lambda, \alpha, \beta) \).

**Proof.** Since

\[
\sum_{k=1}^{\infty} [p + k\lambda][k(1 + \beta) + p(1 - \alpha)]|C_{b}^{k}(k, p)| |a_{p+k}b_{p+k}|
\]

\[
\leq \sum_{k=1}^{\infty} [p + k\lambda][k(1 + \beta) + p(1 - \alpha)]|C_{b}^{k}(k, p)| |a_{p+k}b_{p+k}|
\]

\[
\leq \sum_{k=1}^{\infty} [p + k\lambda][k(1 + \beta) + p(1 - \alpha)]|C_{b}^{k}(k, p)| |a_{p+k}|
\]

\[
\leq p^2(1 - \alpha)
\]

it follows that \( (f \ast g) \in TP_{k,p}^{k^p}(\lambda, \alpha, \beta) \), by the view of Theorem 2. \[ \square \]

**Theorem 7** Let the functions \( f_j(z) (j = 1, 2) \) defined by (9) be in the class \( TP_{k,p}^{k^p}(\lambda, \alpha, \beta) \).

Then the function \( h(z) \) defined by

\[
h(z) = z^p - \sum_{n=2}^{\infty} (a_{1,p+n}^2 + a_{2,p+n}^2)z^{p+n}
\]

is in the class \( TP_{k,p}^{k^p}(\lambda, \xi, \beta) \), where

\[
\delta = 1 - \frac{2p^2(1 - \alpha)^2(1 + \beta)}{[p + \lambda][(1 + \beta) + p(1 - \alpha)]^2|C_{b}^{k}(1, p)| - 2p^3(1 - \alpha)^2}
\]

where \( C_{b}^{k}(1, p) \) is given by (9).
**Proof.** By virtue of Theorem 2, it is sufficient to prove that
\[
\sum_{k=1}^{\infty} \frac{[p + k\lambda][k(1 + \beta) + p(1 - \xi)]C_b^p(k, p)}{p^2(1 - \xi)} [a_{1,p+k}^2 + a_{2,p+k}^2] \leq 1
\] (13)
where \( f_j \in TP_{b,\mu}^{k,p}(\lambda, \alpha, \beta) \) we find from (9) and Theorem 2, that
\[
\sum_{k=1}^{\infty} \left[ \frac{[p + k\lambda][k(1 + \beta) + p(1 - \alpha)]|C_b^p(k, p)|}{p^2(1 - \alpha)} \right]^2 a_{j,p+k}^2 \leq 1, (j = 1, 2)
\] (15)
which would readily yield
\[
\sum_{k=1}^{\infty} \frac{1}{2} \left[ \frac{[p + k\lambda][k(1 + \beta) + p(1 - \alpha)]|C_b^p(k, p)|}{p^2(1 - \alpha)} \right]^2 [a_{1,p+k}^2 + a_{2,p+k}^2] \leq 1.
\] (16)

By comparing (14) and (16), it is easily seen that the inequality (13) will be satisfied if
\[
\frac{[p + k\lambda][k(1 + \beta) + p(1 - \xi)]|C_b^p(k, p)|}{p^2(1 - \xi)} \leq \frac{1}{2} \left[ \frac{[p + k\lambda][k(1 + \beta) + p(1 - \alpha)]|C_b^p(k, p)|}{p^2(1 - \alpha)} \right]^2, \text{ for } k \geq 1.
\]
That is if
\[
\xi = \Psi(k) = 1 - \frac{2p^2(1 - \alpha)^2 k(1 + \beta)}{[p + k\lambda][k(1 + \beta) + p(1 - \alpha)]^2|C_b^p(k, p)| - 2p^3(1 - \alpha)^2}
\] (17)
Since \( \Psi(k) \) is an increasing function of \( k \) (\( k \geq 1 \)). Taking \( k = 1 \) in (17), we have,
\[
\xi = \Psi(1) = 1 - \frac{2p^2(1 - \alpha)^2 (1 + \beta)}{[p + \lambda][(1 + \beta) + p(1 - \alpha)]^2|C_b^p(1, p)| - 2p^3(1 - \alpha)^2}
\]
which completes the proof. \( \square \)

**Concluding Remarks:** In fact, by appropriately selecting the arbitrary sequences given in (10) and (11), suitably specializing the values of \( \mu, \alpha, \beta \) and \( p \) the results presented in this paper would find further applications for the class of \( p \)-valent functions stated in Examples 1 to 4 in Section 1.
References


*Received: April 3, 2011*