Contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold

Barnali Laha
Jadavpur University
Department of Mathematics
Kolkata-700032, India
e-mail: barnali.laha87@gmail.com

Bandana Das
Jadavpur University
Department of Mathematics
Kolkata-700032, India
e-mail: badan06@yahoo.co.in

Arindam Bhattacharyya
Jadavpur University
Department of Mathematics
Kolkata-700032, India
e-mail: bhattchar1968@yahoo.co.in

Abstract. In this paper we prove some properties of the indefinite Lorentzian para-Sasakian manifolds. Section 1 is introductory. In Section 2 we define \(D\)-totally geodesic and \(D^\perp\)-totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold and deduce some results concerning such a manifold. In Section 3 we state and prove some results on mixed totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold. Finally, in Section 4 we obtain a result on the anti-invariant distribution of totally umbilic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold.

1 Introduction

Many valuable and essential results were given on differential geometry with contact and almost contact structure. In 1970 the geometry of cosymplectic

2010 Mathematics Subject Classification: 53C15, 53C21, 53C25, 53C50
Key words and phrases: contact CR-submanifold, totally umbilic submanifold, indefinite Lorentzian para-Sasakian manifold, \(D\)-totally geodesic, \(D^\perp\)-totally geodesic, mixed totally geodesic, anti invariant distribution \(D^\perp\)
manifold was studied by G. D. Ludden [14]. After them, in 1973 and 1974, B. Y. Chen and K. Ogive introduced the geometry of submanifolds and totally real submanifolds in [8], [17], [7]. Then K. Ogive expressed the differential geometry of Kaehler submanifolds in [17]. In 1976 contact manifolds in Riemannian geometry were discussed by D. E. Blair [5]. Later on, A. Bejancu discussed CR-submanifolds of a Kaehler manifold [1], [2], [4], and then, K. Yano and M. Kon gave the notion of invariant and anti invariant submanifold in [13] and [21]. M. Kobayashi studied CR-submanifolds of a Sasakian manifold in 1981 [12]. New classes of almost contact metric structures and normal contact manifold in [18], [6] were studied by J. A. Oubina, C. Calin and I. Mihai. A. Bejancu and K. L. Duggal introduced (ε)-Sasakian manifolds. Lightlike submanifold of semi Riemannian manifolds was introduced by K. L. Duggal and A. Bejancu [10], [9]. In 2003 and 2007, lightlike submanifolds and hypersurfaces of indefinite Sasakian manifolds were introduced [11]. Lastly, LP-Sasakian manifolds were studied by many authors in [15], [16], [19], [20].

In this paper we define \(D\)-totally and \(D^\perp\)-totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold and prove some interesting results.

An \(n\)-dimensional differentiable manifold is called indefinite Lorentzian para-Sasakian manifold if the following conditions hold

\[
\begin{align*}
\phi^2X &= X + \eta(X)\xi, & \eta \circ \phi &= 0, & \phi \xi &= 0, & \eta(\xi) &= 1, \\
\tilde{g}(\phi X, \phi Y) &= \tilde{g}(X, Y) - \varepsilon \eta(X)\eta(Y), \\
\tilde{g}(X, \xi) &= \varepsilon \eta(X),
\end{align*}
\]

for all vector fields \(X, Y\) on \(\tilde{M}\) [5] and where \(\varepsilon\) is 1 or \(-1\) according to \(\xi\) is space-like or time-like vector field.

An indefinite almost metric structure \((\phi, \xi, \eta, \tilde{g})\) is called an indefinite Lorentzian para-Sasakian manifold if

\[
(\tilde{\nabla}_X\phi)Y = g(X, Y)\xi + \varepsilon \eta(Y)X + 2\varepsilon \eta(X)\eta(Y)\xi,
\]

where \(\tilde{\nabla}\) is the Levi-Civita \((L - C)\) connection for a semi-Riemannian metric \(\tilde{g}\). Also we have

\[
\tilde{\nabla}_X\xi = \varepsilon \phi X,
\]

where \(X \in T\tilde{M}\).

From the definition of contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold we have
Definition 1 An \( n \)-dimensional Riemannian submanifold \( M \) of an indefinite Lorentzian para-Sasakian manifold \( \tilde{M} \) is called a contact CR-submanifold if

i) \( \xi \) is tangent to \( M \),

ii) there exists on \( M \) a differentiable distribution \( D : x \mapsto D_x \subset T_x(M) \), such that \( D_x \) is invariant under \( \phi \); i.e., \( \phi D_x \subset D_x \), for each \( x \in M \) and the orthogonal complementary distribution \( D^\perp : x \mapsto D^\perp_x \subset T^\perp_x(M) \) of the distribution \( D \) on \( M \) is totally real; i.e., \( \phi D^\perp_x \subset T^\perp_x(M) \), where \( T_x(M) \) and \( T^\perp_x(M) \) are the tangent space and the normal space of \( M \) at \( x \).

\( D \) (resp. \( D^\perp \)) is the horizontal (resp. vertical) distribution. The contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold is called \( \xi \)-horizontal (resp. \( \xi \)-vertical) if \( \xi_x \in D_x \) (resp. \( \xi_x \in D^\perp_x \)) for each \( x \in M \) by [12].

The Gauss and Weingarten formulae are as follows

\[ \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \]
\[ \tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \]

for any \( X, Y \in TM \) and \( N \in T^\perp M \), where \( \nabla^\perp \) is the connection on the normal bundle \( T^\perp M \), \( h \) is the second fundamental form and \( A_N \) is the Weingarten map associated with \( N \) via

\[ g(A_N X, Y) = g(h(X, Y), N). \]

The equation of Gauss is given by

\[ \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)), \]

where \( \tilde{R} \) (resp. \( R \)) is the curvature tensor of \( \tilde{M} \) (resp. \( M \)).

For any \( x \in M \), \( X \in T_x M \) and \( N \in T^\perp_x M \), we write

\[ X = PX + QX \]
\[ \phi N = BN + CN, \]

where \( PX \) (resp. \( BN \)) denotes the tangential part of \( X \) (resp. \( \phi N \)) and \( QX \) (resp. \( CN \)) denotes the normal part of \( X \) (resp. \( \phi N \)) respectively.
Using (6), (7), (10), (11) in (4) after a brief calculation we obtain on comparing the horizontal, vertical and normal parts

\[ P\nabla_X\phi PY - PA\phi_QYX = \phi P\nabla_XY + g(PX, Y)\xi + \epsilon\eta(Y)PX + 2\epsilon\eta(Y)\eta(X), \]  
\[ Q\nabla_X\phi PY + QA\phi_QYX = Bh(X, Y) + g(QX, Y)\xi + \epsilon\eta(Y)QX, \]  
\[ h(X, \phi PY) + \nabla_X\phi QY = \phi Q\nabla_XY + Ch(X, Y). \]

From (5) we have

\[ \nabla_X\xi = \epsilon\phi PX, \]  
\[ h(X, \xi) = \epsilon\phi QX. \]

Also we have

\[ h(X, \xi) = 0 \quad \text{if} \quad X \in D, \]  
\[ \nabla_X\xi = 0, \]  
\[ h(\xi, \xi) = 0, \]  
\[ A_N\xi \in D^\perp. \]

2 D-totally geodesic and D-$^\perp$-totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold

First we define the D-totally (resp. D-$^\perp$-totally) geodesic contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold.

**Definition 2** A contact CR-submanifold $M$ of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$ is called D-totally geodesic (resp. D-$^\perp$-totally geodesic) if $h(X, Y) = 0$, $\forall X, Y \in D$ (resp. $X, Y \in D^\perp$).

From the above definition, the following propositions follow immediately.

**Proposition 1** Let $M$ be a contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold. Then $M$ is a D-totally geodesic if and only if $A_NX \in D^\perp$ for each $X \in D$ and $N$ a normal vector field to $M$.

**Proof.** Let $M$ be D-totally geodesic. Then from (8) we get

\[ g(h(X, Y), N) = g(A_NX, Y) = 0. \]
Contact CR-submanifolds of indefinite Lorentzian para-Sasakian manifold

So if

\[ h(X, Y) = 0, \quad \forall \ X, Y \in D \]

i.e.,

\[ A_N X \in D^\perp. \]

Conversely, let \( A_N X \in D^\perp \). Then for \( X, Y \in D \) we can obtain

\[ g(A_N X, Y) = 0 = g(h(X, Y), N) \]

i.e.,

\[ h(X, Y) = 0 \]

\( \forall \ X, Y \in D \), which implies that \( M \) is \( D \)-totally geodesic. Thus our proof is complete. \( \square \)

**Proposition 2** Let \( M \) be a contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold \( \tilde{M} \). Then \( M \) is \( D^\perp \)-totally geodesic if and only if \( A_N X \in D \) for each \( X \in D^\perp \) and \( N \) a normal vector field to \( M \).

**Proof.** The proof follows immediately from the above proposition. \( \square \)

Concerning the integrability of the horizontal distribution \( D \) and vertical distribution \( D^\perp \) on \( M \), we can state the following theorem:

**Theorem 1** Let \( M \) be a contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold. If \( M \) is \( \xi \)-horizontal, then the distribution \( D \) is integrable iff

\[ h(X, \phi Y) = h(\phi X, Y) \quad (21) \]

\( \forall \ X, Y \in D \). If \( M \) is \( \xi \)-vertical then the distribution \( D^\perp \) is integrable iff

\[ A_{\phi X} Y - A_{\phi Y} X = \epsilon [\eta(Y)X - \eta(X)Y] \quad (22) \]

\( \forall \ X, Y \in D^\perp \).

**Proof.** If \( M \) is \( \xi \)-horizontal, then using \((14)\) we get

\[ h(X, \phi Y) = \phi Q \nabla_X Y + Ch(X, Y) \]

\( \forall \ X, Y \in D \). Therefore \([X, Y] \in D \) iff \( h(X, \phi Y) = h(Y, \phi X) \)

Hence, if \( M \) is \( \xi \)-horizontal, \([X, Y] \in D \) iff \( h(X, \phi Y) = h(\phi X, Y) \).
Again using (14) we get
\[ \nabla_{\xi} \phi Y = \text{Ch}(X, Y) + \phi Q \nabla_{\chi} Y \] (23)
for \( X, Y \in D^\perp \).

After some calculations we see that
\[ \tilde{\nabla}_{\chi} \phi Y = g(X, Y)\xi + \epsilon \eta(Y)X + 2\epsilon \eta(Y)\eta(X)\xi + \phi P \nabla_{\chi} Y \]
\[ + \phi Q \nabla_{\chi} Y + B h(X, Y) + \text{Ch}(X, Y). \] (24)

Again from (7) and (24) we get
\[ \nabla_{\chi} \phi Y = A_{\phi Y} X + g(X, Y)\xi + \epsilon \eta(Y)X + 2\epsilon \eta(Y)\eta(X)\xi, \]
\[ + \phi P \nabla_{\chi} Y + \phi Q \nabla_{\chi} Y + B h(X, Y) + \text{Ch}(X, Y) \] (25)
for \( X, Y \in D^\perp \). From (24) and (25) we can write
\[ \phi P \nabla_{\chi} Y = -A_{\phi Y} X - g(X, Y)\xi - \epsilon \eta(Y)X - 2\epsilon \eta(Y)\eta(X)\xi - B h(X, Y). \] (26)

Interchanging \( X \) and \( Y \) in (26) we get
\[ \phi P \nabla_{\gamma} Y = -A_{\phi X} Y - g(X, Y)\xi - \epsilon \eta(X)Y - 2\epsilon \eta(Y)\eta(X)\xi - B h(X, Y). \] (27)

Subtracting (27) from (26) we have
\[ \phi P [X, Y] = -A_{\phi Y} X + A_{\phi X} Y - \epsilon \eta(Y)X + \epsilon \eta(X)Y. \] (28)

Now since \( M \) is \( \xi \)-vertical, \( [X, Y] \in D^\perp \) iff
\[ A_{\phi X} Y - A_{\phi Y} X = \epsilon [\eta(Y)X - \eta(X)Y]. \]

So the proof is complete. \( \square \)

D-umbilic (resp. \( D^\perp \)-umbilic) contact CR-submanifold of indefinite Lorentzian para-Sasakian manifold is defined as follows:

**Definition 3** A contact CR-submanifold \( M \) of an indefinite Lorentzian para-Sasakian manifold is said to be D-umbilic (resp. \( D^\perp \)-umbilic) if \( h(X, Y) = g(X, Y)L \) holds for all \( X, Y \in D \) (resp. \( X, Y \in D^\perp \)), \( L \) being some normal vector field.
Contact CR-submanifolds of indefinite Lorentzian para-Sasakian manifold

In view of the above definition we state and prove the following proposition:

**Proposition 3** Suppose $M$ is a $D$-umbilic contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$. If $M$ is $\xi$-horizontal (resp. $\xi$-vertical) then $M$ is $D$-totally geodesic (resp. $D^{\perp}$-totally geodesic).

**Proof.** Consider $M$ as $D$-umbilic $\xi$-horizontal contact CR-submanifold. Then we have from Definition 3

$$h(X, Y) = g(X, Y)L \quad \forall \ X, Y \in D,$$

$L$ being some normal vector field on $M$. By putting $X = Y = \xi$ and using (19) we have

$$h(\xi, \xi) = g(\xi, \xi)L$$

i.e.

$$L = 0,$$

and consequently we get $h(X, Y) = 0$, which proves that $M$ is $D$-totally geodesic.

Similarly, it can be easily shown that if $M$ is $D^{\perp}$-umbilic $\xi$-vertical contact CR-submanifold then it is $D^{\perp}$-totally geodesic. \[\square\]

3 Mixed totally geodesic contact CR-submanifolds of indefinite Lorentzian para-Sasakian manifold

In this section we define mixed totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold (followed [12]).

**Definition 4** A contact CR-submanifold $M$ of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$ is said to be mixed totally geodesic if $h(X, Y) = 0 \ \forall \ X \in D$ and $Y \in D^{\perp}$.

Then we extract the following lemma and theorem

**Lemma 1** Let $M$ be a contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold. Then $M$ is mixed totally geodesic iff

$$A_{\xi}X \in D, \quad \forall \ X \in D, \quad \text{and} \quad \forall \ \text{normal vector field } N, \quad (29)$$

$$A_{\xi}X \in D^{\perp}, \quad \forall \ X \in D^{\perp} \quad \text{and} \quad \forall \ \text{normal vector field } N. \quad (30)$$
Proof. If $M$ is mixed totally geodesic, then from (8), we get
$$h(X, Y) = 0,$$
i.e., iff $A_N X \in D, \forall X \in D$ and $\forall$ normal vector field $N$. Conversely, if $M$ is mixed totally geodesic, then using (8) we easily observe that $A_N X \in D^\perp, \forall X \in D^\perp$ and $\forall$ normal vector field $N$.
Hence the lemma is proved. \hfill $\Box$

Using condition (29) we obtain the following theorem

**Theorem 2** If $M$ is a mixed totally geodesic contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold, then

$$A_{\phi N} X = -\phi A_N X,$$  \hspace{1cm} (31)

$$\nabla_X \phi N = \phi \nabla^\perp_X N$$  \hspace{1cm} (32)

$\forall X \in D$ and $\forall$ normal vector field $N$.

**Proof.** We get from (29), (6), (7) and after having some calculations we derive

$$\nabla_X \phi N = \phi \nabla^\perp_X N - \phi A_N X,$$  \hspace{1cm} (33)

$$\nabla_X \phi N = -A_{\phi N} X + \nabla^\perp_X \phi N.$$  \hspace{1cm} (34)

Comparing the above two equations we have the required theorem. Hence the proof follows. \hfill $\Box$

Again we have the following definition

**Definition 5** A contact CR-submanifold $M$ of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$ is called foliate contact CR-submanifold $\tilde{M}$ if $D$ is involute. If $M$ is a foliate $\xi$-horizontal contact CR-submanifold, we know from [3]

$$h(\phi X, \phi Y) = h(\phi^2 X, Y) = -h(X, Y).$$  \hspace{1cm} (35)

Considering the above definition we give the following proposition.

**Proposition 4** If $M$ is a foliate $\xi$-horizontal mixed totally geodesic contact CR-submanifold $M$ of an indefinite Lorentzian para-Sasakian manifold, then

$$\phi A_N X = A_N \phi X$$  \hspace{1cm} (36)

for all $X \in D$ and normal vector field $N$. \\
Proof. From (21) and (8) we compute the following:

$$g(h(X, \phi Y), N) = g(\phi N X, Y),$$

i.e.

$$g(h(\phi X, Y), N) = g(N \phi X, Y).$$

Therefore

$$\phi N X = N \phi X.$$

Hence the proof follows. □

4 Anti-invariant distribution $D^\perp$ on totally umbilical contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold

Here we consider a contact CR-submanifold $M$ of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$. Then we establish the following theorem.

Theorem 3 Let $M$ be a totally umbilical contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$. Then the anti invariant distribution $D^\perp$ is one dimensional, i.e. $\dim D^\perp = 1$.

Proof. For an indefinite Lorentzian para-Sasakian structure we have

$$(\tilde{\nabla}_Z \phi) W = g(Z, W) \xi + \epsilon \eta(W) Z + 2\epsilon \eta(W) \eta(Z) \xi. \quad (37)$$

Also by the covariant derivative of tensor fields (for any $Z, W \in \Gamma(D^\perp)$) we know

$$\tilde{\nabla}_Z \phi W = (\tilde{\nabla}_Z \phi) W + \phi \tilde{\nabla}_Z W. \quad (38)$$

Using (37), (38), (6), (7) and (4) we obtain

$$\nabla^\perp Z \phi W - g(H, \phi W) Z = \phi [\nabla Z W + g(Z, W) H] + g(Z, W) \xi$$

$$+ \epsilon \eta(W) Z + 2\epsilon \eta(W) \eta(Z) \xi \quad (39)$$

for any $Z, W \in \Gamma(D^\perp)$.

Taking the inner product with $Z \in \Gamma(D^\perp)$ in (39) we obtain

$$-g(H, \phi W) ||Z||^2 = g(Z, W) g(\phi H, Z) + \epsilon \eta(W) ||Z||^2 + g(Z, W) g(\xi, Z)$$

$$+ 2\eta(W) \eta(Z) g(Z, \xi). \quad (40)$$
Using (2) after a brief calculation we have
\[ g(H, \phi W) = -\frac{g(Z, W)g(\phi H, Z)}{||Z||^2} - \frac{g(Z, W)g(\xi, Z)}{||Z||^2} \]
\[ - \epsilon g(W, \xi) - 2\frac{g(Z, \xi)^2 g(W, \xi)}{||Z||^2}. \]  \hfill (41)

Interchanging \( Z \) and \( W \) we have
\[ g(H, \phi Z) = -\frac{g(Z, W)g(\phi H, W)}{||W||^2} - \frac{g(Z, W)g(\xi, W)}{||W||^2} \]
\[ - \epsilon g(Z, \xi) - 2\frac{g(W, \xi)^2 g(Z, \xi)}{||W||^2}. \]  \hfill (42)

Substituting (41) in (40) and simplifying we get
\[ g(H, \phi W) \left[ 1 - \frac{g(Z, W)^2}{||Z||^2||W||^2} \right] - \frac{g(Z, W)}{||Z||^2} \left[ \frac{g(Z, W)g(\xi, W)}{||W||^2} - g(Z, \xi) \right] \]
\[ - \epsilon \left[ \frac{g(Z, W)g(\xi, Z)}{||Z||^2} - g(W, \xi) \right] \]
\[ - 2g(z, \xi)g(W, \xi) \left[ \frac{g(Z, W)g(W, \xi)}{||W||^2||Z||^2} - \frac{g(Z, W)}{||Z||^2} \right] = 0. \]  \hfill (43)

The equation (43) has a solution if \( Z \parallel W \), i.e. \( \dim D^\perp = 1 \).
Hence the theorem is proved. \( \Box \)

**Example 1** Let \( \mathbb{R}^3 \) be a 3-dimensional Euclidean space with rectangular co-ordinates \((x, y, z)\). In \( \mathbb{R}^3 \) we define
\[ \eta = -dz - ydx \]
\[ \xi = \frac{\partial}{\partial z} \]
\[ \phi(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}, \quad \phi(\frac{\partial}{\partial y}) = \frac{\partial}{\partial x} - y \frac{\partial}{\partial z}, \quad \phi(\frac{\partial}{\partial z}) = 0. \]

The Lorentzian metric \( g \) is defined by the matrix:
\[
\begin{pmatrix}
-\epsilon y^2 & 0 & cy \\
0 & 0 & 0 \\
cy & 0 & -\epsilon 
\end{pmatrix}
\]
Then it can be easily seen that $(\phi, \xi, \eta, g)$ forms an indefinite Lorentzian para-Sasakian structure in $\mathbb{R}^3$ and the above results can be verified for this example.

Acknowledgement

This work is sponsored by UGC – BSR, UGC, India.

References


Received: 3 September 2013